VIRTUALLY GEOMETRICALLY FINITE MAPPING CLASS GROUPS OF 3-MANIFOLDS

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Abstract

If M is a compact orientable irreducible sufficiently large 3-manifold, then the mapping class group $\mathscr{H}(M)$ contains a subgroup of finite index which is the fundamental group of a finite aspherical CW-complex. If in addition the boundary of M is incompressible, then $\mathscr{H}(M)$ contains a subgroup of finite index which is a duality group. For many cases, the virtual cohomological dimension of $\mathscr{H}(M)$ is calculated.

0. Introduction

The 3-manifolds considered in this work are compact, orientable, irreducible, and sufficiently large. If in addition such a manifold is closed or has incompressible boundary, we say that it is *Haken*.

We are concerned with the mapping class groups of these 3-manifolds. We consider two "finiteness" properties that a group Γ may enjoy:

(1) Γ is of type FL; that is, there is a finite resolution of the trivial Γ-module $\mathbb{Z}$ by finitely generated free $\mathbb{Z}\Gamma$-modules.

(2) Γ is a duality group (over $\mathbb{Z}$); that is, there is a (right) $\mathbb{Z}\Gamma$-module C such that for some nonnegative integer n there are natural isomorphisms $H^k(\Gamma; A) \cong H_{n-k}(\Gamma; C \otimes_{\mathbb{Z}} A)$ for all k and all $\mathbb{Z}\Gamma$-modules A.

In (2), n is called the dimension of the duality group. For (1) our reference is [35] and for (2) it is [2] (see also [1]).

When Γ is finitely presented, properties (1) and (2) have topological interpretations which we discuss in §1 along with other preliminaries. Both properties easily imply that the cohomological dimension of Γ is finite. For duality groups the cohomological dimension is equal to the dimension as a duality group.

We say that Γ is of type VFL (respectively, a virtual duality group) if there is a subgroup of finite index in Γ which is of type FL (respectively, which is a duality group). The *virtual cohomological dimension* (see [35]) is

the cohomological dimension of any torsion-free subgroup of finite index, if such a subgroup exists. This is independent of the choice of subgroup.

It has long been known that the mapping class groups of 2-manifolds (of finite type) are finitely presented; this was first proved by McCool [27]. Work of other authors, notably Harvey [16], [17] later showed that they are of type VFL, and in recent work Harer [14] has proved that they are virtual duality groups.

In the present work these results are extended to 3-manifolds. Precisely, we prove

Theorem. *Let M be a compact orientable irreducible sufficiently large 3-manifold. Then the mapping class group $\mathscr{H}(M)$ is finitely presented and is of type VFL. If M is Haken, then $\mathscr{H}(M)$ is a virtual duality group.*

The first seven sections of this paper prove the Theorem as follows.

The characteristic submanifold theory due to Johannson [21] and Jaco and Shalen [20] shows that Haken 3-manifolds consist of invariant (up to isotopy) pieces which are either Seifert fibered, I-fibered, or "simple"; the latter have finite mapping class groups [21] and are negligible when considering virtual properties of $\mathscr{H}(M)$. For a Seifert fibered piece Σ with orbit surface F, the mapping class group is studied in §3. Apart from a few exceptional cases which can be handled explicitly, the orientation-preserving mapping class group $\mathscr{H}_+(\Sigma)$ is isomorphic to the group of orientation-preserving fiber-preserving mapping classes $\mathscr{H}_+^f(\Sigma)$. From [21, Propositions 25.2 and 25.3], excepting some more cases, there is an exact sequence

$$1 \to H_1(F, \partial F) \to \mathscr{H}_+^f(\Sigma) \to \mathscr{H}^*(F) \to 1$$

in which $\mathscr{H}^*(F)$ is a subgroup of finite index in $\mathscr{H}(F')$, where F' is the result of removing from F the points which correspond to exceptional orbits of Σ. The kernel $H_1(F, \partial F)$ is isomorphic to the group of "vertical" mapping classes that map each fiber to itself. Work of Harer [14], which we extend in §2 to nonorientable 2-manifolds, shows that $\mathscr{H}^*(F)$ is a virtual duality group, and the intersection of the vertical mapping classes with a certain subgroup of finite index in $\mathscr{H}(\Sigma)$ is a finitely generated free abelian group. This proves the Theorem for the Seifert fibered case. For I-bundles, the mapping class group is essentially the same as the mapping class group of the orbit surface, and the Theorem follows from the 2-dimensional version.

For the general case of incompressible boundary, considered in §4, there is a subgroup of finite index in $\mathscr{H}(M)$ which maps onto the product of

(certain subgroups of) the mapping class groups of the components of the characteristic submanifold of M, with kernel the finitely generated abelian subgroup generated by Dehn twists about the components of the frontier of the characteristic submanifold. Somewhat surprisingly, this kernel can contain torsion, and some effort is required to find a subgroup of finite index in $\mathscr{H}(M)$ that avoids this torsion. These ideas combine to yield the Theorem in the Haken case.

In the case of compressible boundary, there is a third kind of characteristic piece called a product-with-handles, studied in [3], [28]. For a product-with-handles V, there is a simplicial complex L in which the vertices are the isotopy classes of essential compressing discs in V, and a collection of vertices spans a simplex if and only if the isotopy classes can be represented by a collection of discs in V which are pairwise disjoint. We prove in §5 that L is a finite-dimensional contractible complex admitting a simplicial action of $\mathscr{H}(V)$ with finite quotient. The result of cutting a product-with-handles along a set of compressing discs is a collection of products-with-handles of lower complexity; this enables us in §6 to analyze the stabilizers of simplices in L inductively, obtaining enough information to establish that $\mathscr{H}(V)$ is finitely-presented and virtually of type FL. The proof of the Theorem for manifolds with compressible boundary is given in §7. It is based on induction on the number of compressible boundary components, with the induction starting from the Haken case.

For the cases when the boundary of M is compressible I do not know in general whether $\mathscr{H}(M)$ is a virtual duality group. In §8, we use very special facts about the genus 2 orientable handlebody V_2 to prove that $\mathscr{H}(V_2)$ is a virtual duality group of dimension 3.

In §9, we calculate the virtual cohomological dimension of the mapping class groups for the Haken case, and give bounds for it when M is a handlebody or a product-with-handles.

It follows from work of Johannson [21] and Hemion [18] (see the discussion in [40]) that $\mathscr{H}(M)$ is finitely presented in the Haken case. For the case of compressible boundary, finite generation was proved in [28], and finite presentation was proved by R. Kramer (unpublished) for handlebodies and by P. Grasse [10], [11] in general. The present work is an outgrowth of the discussions which led to the latter.

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1. Preliminaries

Let A_1, A_2, $\cdots$, A_n, B be possibly empty subspaces of a PL manifold M. By the mapping class group $\mathscr{H}(M, A_1, A_2, \cdots, A_n \, rel \, B)$ we mean the group of path components of the group of homeomorphisms

$$\text{Homeo}\,(M, A_1, A_2, \cdots, A_n \, \text{rel}\, B) = \{h \, | \, h\,(A_i) = A_i$$
$$\text{for } 1 \le i \le n \text{ and } h|_B = \text{id}_B\}\,.$$

A "plus" subscript, as in $\mathscr{H}_+(M)$, indicates the subgroup of orientation-preserving classes if M is orientable, while for nonorientable M, $\mathscr{H}(M) = \mathscr{H}_+(M)$.

There seems to be some discrepancy in terminology of the literature, with many (perhaps most) authors referring to $\mathscr{H}(M)$ as the homeotopy group and to the subgroup of orientation-preserving classes as the mapping class group. We will sometimes use the (redundant, for us) terminology "full mapping class group" when the presence of orientation-reversing classes may have significance.

It is known that every 2- or 3-dimensional manifold admits unique PL and differentiable structures, and moreover that the inclusions from the diffeomorphism group into the group of PL homeomorphisms, and from the latter into the group of all homeomorphisms, are homotopy equivalences. This allows us to move rather freely between the three categories as dictated by convenience, and we do so without explicit reselection of notation.

For any manifold M, there is a homomorphism Φ from $\mathscr{H}(M)$ to the group $\text{Out}(\pi_1(M))$ of outer automorphisms of the fundamental group of M, defined by sending the isotopy class of h to the outer automorphism represented by the induced automorphism $h_\#\colon \pi_1(M) \to \pi_1(M)$. For low-dimensional manifolds, Φ gives a great deal of information about $\mathscr{H}(M)$. For example, if M is a closed 2-manifold other than the 2-sphere, it is well known that Φ is an isomorphism. For *closed* Haken 3-manifolds, Waldhausen [39] proved that Φ is an isomorphism, while for Haken 3-manifolds with boundary, the kernel of Φ is nontrivial only when M is an I-bundle, in which case it is of order 2, generated by reflection in the I-fibers, and the image is as large as the fundamental group allows—it is the outer automorphisms that take the image of the fundamental group of each boundary component to a conjugate of the image of the fundamental group of a boundary component (these are called the outer automorphisms that preserve the peripheral structure).

Although these results are quite useful, it is difficult to extract information about $\mathrm{Out}(\pi_1(M))$ by algebraic methods. For example, it is far from evident algebraically that $\mathrm{Out}(\pi_1(M))$ is even finitely generated. Thus, our results provide otherwise inaccessible information about $\mathrm{Out}(\pi_1(M))$.

In the introduction, we gave algebraic definitions for the terms FL and duality group. There are useful topological interpretations for these properties.

Theorem 1.1. *Let Γ be a finitely presented group.*

(a) *Γ is of type FL if and only if there is a finite CW-complex which is a $K(\Gamma, 1)$.*

(b) *Suppose X is a compact m-dimensional manifold with nonempty boundary which is a $K(\Gamma, 1)$. Let $\tilde{X}$ denote the universal cover of X. Then Γ is a duality group if and only if for some q the (reduced) homology groups $H_i(\partial \tilde{X})$ are zero for $i \neq q$ and $H_q(\partial \tilde{X})$ is torsion-free, in which case the dimension of Γ is $m - q - 1$.*

Part (a) is proved in [35, Proposition 10] and part (b) in [2].

A group which admits a finite $K(\Gamma, 1)$ complex is sometimes called a *geometrically finite* group.

The following examples will appear frequently.

Examples 1.2. (a) If G is finite, then G is a virtual duality group of type VFL of dimension zero.

(b) If G is free and finitely generated, then G is a duality group of type FL of dimension 1.

(c) If $G \cong \mathbb{Z}^k$, then G is a duality group of type FL of dimension k.

(d) If $G \cong \mathrm{GL}(k, \mathbb{Z})$, then G is a virtual duality group of type VFL of dimension $k(k-1)/2$.

Proof. (a) is obvious since the trivial group has finite index in G. Parts (b) and (c) follow by applying Theorem 1.1 when X is a 2-disc with holes and the product of k-dimensional torus and an interval. Part (d) is from [4].

We will make frequent use of the following facts.

Lemma 1.3. *Let Γ' be a subgroup of finite index in Γ.*

(a) *If Γ is of type FL, then so is Γ'.*

(b) *If Γ is a duality group, then so is Γ'.*

(c) *If Γ' is a duality group and Γ is torsion free, then Γ is a duality group.*

Part (a) is [35, Proposition 5(c)], and (b) and (c) are [2, Theorems 3.2 and 3.3]. To my knowledge the analogue of (c) for groups of type FL is still unknown (see [35]).

We will frequently exploit the behavior of groups of type VFL and virtual duality groups under extensions.

Lemma 1.4. *Let* $1 \to A \to B \to C \to 1$ *be an exact sequence of groups.*

(a) *If* A *and* C *are finitely presented, then so is* B .

(b) *If* A *is of type FL and* C *is of type VFL, then* B *is of type VFL.*

(c) *If* A *is a duality group and* C *is a virtual duality group, then* B *is a virtual duality group, and* $\dim(B) = \dim(A) + \dim(C)$.

Proof. Result (a) is well known. For (b) and (c), by pulling back the extension to a finite index subgroup of C, we may assume that C is of type FL or is a duality group. Then, the result follows from [35, Proposition 6(b); 2, Theorem 3.5].

Parts (b) and (c) of Lemma 1.4 do not extend to the case when A has the virtual property. Examples may be found in [31].

2. Mapping class groups of 2-manifolds

Let F be a connected 2-manifold with r boundary circles and s punctures. If F is orientable with orientable genus g, then the group of orientation-preserving mapping classes that take each puncture to itself and restrict to the identity on each component of ∂F is denoted by $\Gamma^s_{g,r}$. If F is nonorientable with nonorientable genus g (i.e, F is obtained from a connected sum of g projective planes by removing r open discs and s points), then the group of mapping classes that take each puncture to itself and restrict to the identity on each component of ∂F is denoted by $\Lambda^s_{g,r}$. These have finite index in $\mathcal{H}(F \text{ rel } \partial F)$. It is sometimes convenient to regard some of the punctures as distinguished points fixed by all homeomorphisms and isotopies.

In [14], John Harer proved the following theorem.

Theorem. $\Gamma^s_{g,r}$ *is a finitely presented virtual duality group of type VFL. If* $2g + r + s > 2$, *then the dimension of* $\Gamma^s_{g,r}$ *is* $d(g,r,s)$, *where* $d(g,0,0) = 4g - 5$, $d(g,r,s) = 4g + 2r + s - 4$ *if* $g > 0$ *and* $r + s > 0$, *and* $d(0,r,s) = 2r + s - 3$.

In this section, relying heavily on Harer's ideas, we prove

Theorem 2.1. $\Lambda^s_{g,r}$ *is a finitely presented virtual duality group of type VFL. If* $g + r + s > 2$, *then the dimension of* $\Lambda^s_{g,r}$ *is* $2g - 5$ *if* $r = s = 0$ *and is* $2g + 2r + s - 4$ *if* $r + s > 0$.

If $\chi(N) \geq 0$, then N is either a projective plane or a compact or open Möbius band, whose mapping class groups all have order 1, or a Klein bottle, whose mapping class group has order 4 [25], and Theorem 2.1

is immediate. So in the remainder of this section we will assume that $\chi(N) < 0$.

Since $\Lambda_{g,r}^s$ is an extension of a free abelian group (of rank exactly r when $g+r+s > 2$) by the group $\Lambda_{g,0}^{r+s}$, the proof of Theorem 2.1 is easily reduced to the case $r = 0$. During some portions of the proof, however, it will be necessary to consider bounded surfaces.

To fix notation, assume that N is nonorientable with nonorientable genus g, has s punctures, and r boundary components. Let $p: \tilde{N} \to N$ denote the orientable double cover of N with covering transformation τ. Since each boundary component of N is an orientation-preserving loop, $\tilde{N}$ has $2r$ boundary components, and $2s$ punctures. The (orientable) genus $\tilde{g}$ of $\tilde{N}$ equals $g - 1$. Let $\widetilde{\mathscr{H}}$ denote the full mapping class group of $\tilde{N}$, and $\widetilde{\mathscr{H}}_+$ the subgroup of orientation-preserving classes. Finally, let $\mathscr{H}$ denote the full mapping class group of N. To prove Theorem 2.1, it is necessary to adapt a considerable portion of [14] to the nonorientable case. The discussion will be broken down into three subsections. In §2.1, the mapping class group of N is identified with the stabilizer, under the action of $\widetilde{\mathscr{H}}$ on the Teichmüller space $\mathscr{T}$ of $\tilde{N}$, of the subspace of $\mathscr{T}$ fixed by the covering involution τ. In §2.2, we extend Harer's ideal triangulation of Teichmüller space to the nonorientable case; the fact that the Teichmüller space for the nonorientable case can be identified with the subcomplex fixed by τ acting on the Teichmüller space of the orientable double cover leads to an easy proof of the crucial fact that the addition of the ideal points of this triangulation yields a space which is still contractible. The third and final subsection uses Harvey's bordification of Teichmüller space; a trick allows us to see that the intersection of that bordification (constructed as a subspace of Teichmüller space) with the fixed-point set of τ has compact quotient under the action of its stabilizer, which will allow Theorem 1.1(b) to be applied once the boundary has been analyzed. The analysis of this boundary follows Harer's approach, and completes the third subsection and the proof of Theorem 2.1.

2.1. The imbedding of $\mathscr{H}$ in $\widetilde{\mathscr{H}}_+$. Let $\mathscr{T}_{\tilde{g},2r}^{2s}$ denote the Teichmüller space (= space of all marked complete finite-area hyperbolic structures with totally geodesic boundary, up to isotopy) of $\tilde{N}$. Now $\widetilde{\mathscr{H}}$ acts (on the left) properly discontinuously as isometries (with respect to the Teichmüller metric) on $\mathscr{T}_{\tilde{g},2r}^{2s}$. Although this action is often only considered for orientation-preserving mapping classes, the definition of the action in terms of pulling back hyperbolic metrics (see [9]) applies to orientation-reversing classes. Considering Fenchel-Nielsen coordinates, (see, for example, [15, Chapter 1, §2]) using a τ-equivariant pair-of-pants

decomposition of the orientable double cover shows that the fixed-point
set of $\langle \tau \rangle$ is a Euclidean (in fact, it is totally geodesic) subspace $\mathcal{T}_\tau$ which
can be identified with the Teichmüller space of N, and has dimension
$3\tilde{g} - 3 + 3r + 2s = 3g - 6 + 3r + 2s$.

The first step is to identify $\mathcal{H}$ with the stabilizer of $\mathcal{T}_\tau$ under the action
of $\widetilde{\mathcal{H}}_+$. Let $\widetilde{\mathcal{H}}_\tau$ denote $\{\langle h \rangle \in \widetilde{\mathcal{H}}_+ | \langle h \rangle (\mathcal{T}_\tau) = \mathcal{T}_\tau\}$. Define $\Psi \colon \mathcal{H} \to \widetilde{\mathcal{H}}_+$ by
letting $\Psi(\langle f \rangle)$ be $\langle \tilde{f} \rangle$, where $\tilde{f}$ is the unique orientation-preserving lift
of f to $\tilde{N}$. Since both $\tilde{f}\tau$ and $\tau\tilde{f}$ are orientation-reversing lifts of f,
they are equal. Therefore if $x \in \mathcal{T}_\tau$, we have $\langle \tau \rangle \langle \tilde{f} \rangle x = \langle \tilde{f} \rangle \langle \tau \rangle x = \langle \tilde{f} \rangle x$,
so $\langle \tilde{f} \rangle \in \widetilde{\mathcal{H}}_\tau$. It is clear that Ψ is a homomorphism.

Lemma 2.1.1. $\Psi \colon \mathcal{H} \to \widetilde{\mathcal{H}}_\tau$ *is an isomorphism.*

Proof. Let $\langle h \rangle \in \mathcal{H}$. By isotopy, we may assume that h fixes a base-
point of N and preserves the local orientation there; then the orientation-
preserving lift $\tilde{h}$ will also preserve the basepoint of $\tilde{N}$ (which we assume
to be chosen in the preimage of the basepoint of N). If $\langle \tilde{h} \rangle = \langle 1_{\tilde{N}} \rangle$,
then $\tilde{h}_\# = 1_{\pi_1(\tilde{N})}$ in $\operatorname{Out}(\pi_1(\tilde{N}))$. We may change h by an isotopy to
ensure that $\tilde{h}_\# = 1_{\pi_1(\tilde{N})}$ in $\operatorname{Aut}(\pi_1(\tilde{N}))$. Choose an orientation-reversing
element y of $\pi_1(N)$. Then $h_\#(y) = yz$ for some $z \in p_\#\pi_1(\tilde{N})$. But for
all $g \in p_\#\pi_1(\tilde{N})$, we have $ygy^{-1} = h_\#(ygy^{-1}) = yzgz^{-1}y^{-1}$, hence z is
central in $p_\#\pi_1(\tilde{N})$. Since we are assuming $\chi(N) < 0$, $\pi_1(\tilde{N})$ is centerless
and therefore $z = 1$. Thus $h_\# = 1_{\pi_1(N)}$, so h is isotopic to 1_N, showing
that Ψ is injective.

Now suppose that $g \in \widetilde{\mathcal{H}}_\tau$. Then for all $x \in \mathcal{T}_\tau$, we have $\langle \tau \rangle \langle g \rangle x =
\langle g \rangle x$, so $\langle g^{-1}\tau g \rangle x = x$. Using this information, we will produce a home-
omorphism G of N whose lift $\tilde{G}$ is isotopic to g, thereby showing that
Ψ is surjective. We may regard $\tilde{N}$ in a standard position in $\mathbb{R}^3$, symmet-
ric with respect to the origin, intersecting the x-axis in a finite number of
points and the yz-plane in a set C which is a circle (when g is even) or
two circles (when g is odd) in such a way that τ is the restriction of the
map sending $(a; b, c)$ to $(-a, -b, -c)$, and the "hyperelliptic" involu-
tion ζ is the restriction of a rotation through angle π about the x-axis.
Note that ζ commutes with τ. A hyperbolic structure x on $\tilde{N}$ can be
chosen so that its stabilizer under the action of $\widetilde{\mathcal{H}}$ is the group of order 4
generated by $\langle \tau \rangle$ and $\langle \zeta \rangle$ (in fact, in most cases it can be chosen so that
$\langle \tau \rangle$ generates the stabilizer; in the remaining cases, such as $g = 2$ and
$r = s = 0$, the hyperelliptic symmetry cannot be avoided). Since $\langle g\tau g^{-1} \rangle$
is orientation-reversing and fixes x, it must equal either $\langle \tau \rangle$ or $\langle \tau\zeta \rangle$.

But the fixed-point set of $\tau\zeta$ is C, while the fixed-point set of $g\tau g^{-1}$ is empty, so $\tau\zeta$ and $g\tau g^{-1}$ are not equivalent involutions. By Theorem 6.1 of [37], isotopic involutions of a closed orientable surface must be equivalent. Therefore $\langle g^{-1}\tau g\rangle = \langle\tau\rangle$, so $g_\#^{-1}\tau_\# g_\# = \tau_\#$ in $\text{Out}(\pi_1(\tilde N))$. It follows from the theory of group extensions that there is an automorphism $G_\#$ of $\pi_1(N)$ that restricts to $g_\#$ on the subgroup $p_\#\pi_1(\tilde N)$, but in our case we can easily give a direct construction of $G_\#$ as follows. Choose any $y \in \pi_1(N) - \pi_1(\tilde N)$; then $g_\#^{-1}\mu(y)g_\# = \mu(y)\mu(c)$ for some $c \in \pi_1(\tilde N)$, where $\mu(y)$ denotes conjugation by y. Define $G_\#|_{\pi_1(\tilde N)} = g_\#$ and $G_\#(y) = yg_\#(c^{-1})$. A computation shows that $G_\#$ is an isomorphism. Now $G_\#$ preserves the peripheral structure of N, so is induced by some homeomorphism G of N, and $\tilde G$ induces $g_\#$ so is isotopic to g. This shows that Ψ is surjective and completes the proof of Lemma 2.1.1.

2.2. The complex $B(\Delta)$. Harer's approach makes use of an ideal triangulation of Teichmüller space contained in a complex called $A(\Delta)$. The definition of $A(\Delta)$ given in §1 of [14] applies equally well to nonorientable surfaces, to wit: given a surface S, orientable or not, with s punctures and r boundary components, define a similar complex $B(\Delta)$ as follows. Let $\{p_1, p_2, \cdots, p_s\}$ be a collection of distinguished points in $S - \partial S$. For $\Delta_1 = \{p_1, \cdots, p_m\}$, $m \leq s$, and $\Delta_2 = \{q_1, q_2, \cdots, q_n\}$ a set of points in ∂S, containing at least one point in each boundary component of S, put $\Delta = \Delta_1 \cup \Delta_2$. Assume that Δ is nonempty. Put $P = \{p_{m+1}, \cdots, p_s\}$ and let $S_0 = S - P$ so that S_0 has m distinguished points and $s - m$ punctures.

A *rank k arc-system* is the (ambient) isotopy class $[\alpha_0, \alpha_1, \cdots, \alpha_k]$ of a collection of arcs in S_0 between two points of Δ and loops in S_0 based at points of Δ such that

(1) $\alpha_i \cap \alpha_j \subset \Delta$ for distinct i and j, and

(2) for each component G of the surface obtained by splitting S_0 along $\bigcup\alpha_i$, the Euler characteristic of the double of G along $\partial G - \Delta$ is negative.

A cell complex $B(\Delta)$ is formed by taking a k-simplex $\langle a_0, \cdots, a_k\rangle$ for each rank k arc-system in S and identifying $\langle\beta_0, \cdots, \beta_l\rangle$ as a face of $\langle a_0, \cdots, a_k\rangle$ whenever $\{[\beta_0], \cdots, [\beta_l]\} \subseteq \{[a_0], \cdots, [a_k]\}$. If S is orientable with genus $\tilde g$, then $B(\Delta)$ is $(6\tilde g - 6 + 3r + 2s + m + n - 1)$-dimensional, while if S is nonorientable with genus g, then $B(\Delta)$ is $(3g - 6 + 3r + 2s + m + n - 1)$-dimensional.

The appropriate group $\Gamma_{g,r}^s$ or $\Lambda_{g,r}^s$ acts simplicially on $B(\Delta)$.

An arc-system (i.e., a vertex of $B(\Delta)$) is said to *fill* S if every component of the complement of $(\bigcup\alpha_i)\cup\Delta$ in S_0 is topologically a disc or a disc

punctured once at a point of P. The full subcomplex of $B(\Delta)$ spanned by vertices which do not fill S is denoted by $B(\Delta)_\infty$.

Specializing to our situation, let Δ be a collection of points in N, as described above, and let $\widetilde{\Delta}$ be the preimage of Δ in $\widetilde{N}$. Define $\mathscr{T}^{2s}_{\tilde{g},2r}(\widetilde{\Delta})$ to be the space of all pairs (x, λ), where x is a point in $\mathscr{T}^{2s}_{\tilde{g},2r}$ and λ is the projective class of a collection of positive weights on the $2m + 2n$ points of $\widetilde{\Delta}$, topologized as the product of $\mathscr{T}^{2s}_{\tilde{g},2r}$ with an open simplex of dimension $2m + 2n - 1$. The full mapping class group $\widetilde{\mathscr{H}}$ of $\widetilde{N}$ acts properly discontinuously on $\mathscr{T}^{2s}_{\tilde{g},2r}(\widetilde{\Delta})$ using the standard action in the first factor and the permutation action in the second.

Let $B = B(\Delta)$ and $A = B(\widetilde{\Delta})$. There is an imbedding of B into A defined by sending an arc-system to its preimage. Notice that the preimage of A_∞ is precisely B_∞. Obviously, the image of B is contained in the subcomplex fixed by τ; conversely, if an arc-system is preserved up to isotopy by τ, then it is easy to show that it is isotopic to a τ-invariant arc-system (deform the arcs to have minimal length in a τ-invariant hyperbolic structure), hence the image of B is precisely the subcomplex fixed by τ.

We have the following result from [14].

Theorem 2.2.1. *For $\widetilde{N}$, there is a $\widetilde{\mathscr{H}}$-equivariant homeomorphism from $\mathscr{T}^{2s}_{\tilde{g},2r}(\widetilde{\Delta})$ to $A - A_\infty$.*

The proof is given in [14] (there, the restricted mapping class group $\Gamma^{2s}_{\tilde{g},2r}$ is used, but all constructions are equivariant with respect to the action of the full mapping class group).

Because of the equivariance, the product of the fixed-point set of τ acting on the Teichmüller space with the open $(m + n - 1)$-simplex of invariant projective weights is carried to $B - B_\infty$, yielding

Corollary 2.2.2. $B - B_\infty$ *is contractible.*

In order to prove that B is contractible, we make use of another of Harer's results. In the first barycentric subdivision A^0 of A, define Y^0 to be the full subcomplex of A^0 whose vertices are families $[a_0, \cdots, a_k]$ which fill up S, i.e., which have no face in A_∞. As observed in §2 of [14], Y^0 is the first barycentric subdivision of a regular cell complex Y. This is a subcomplex of the dual cell complex to A; it has a vertex for each maximal arc-system which fills up S, an edge for each arc-system which can be completed to a maximal family by the addition of one arc (necessarily in two ways, yielding maximal families which are the endpoints of the edge), and so on. From Theorem 2.1 of [14] we have

Theorem 2.2.3. *There exists an equivariant deformation retraction from A^0 to Y^0, which provides an equivariant homotopy equivalence between $A^0 - A^0_\infty$ and Y^0.*

In [14], the equivariance is only stated for the action of $\Gamma^{2s}_{\check{g},2r}$, but the proof shows equivariance for the action of the full mapping class group. Consequently, in our particular case, where (B, B_∞) is imbedded naturally as the fixed-point set under the action of τ on (A, A_∞), the deformation carries B onto $Y \cap B$. This lies in $B - B_\infty$, which is contractible by Corollary 2.2.2. Therefore we have

Corollary 2.2.4. $B(\Delta)$ *is contractible.*

2.3. The bordification of Teichmüller space. Assume for now that $r = 0$ (and hence $n = 0$). Let $\Gamma = \Gamma^{2s}_{\check{g},0}$ and $\Gamma_\tau = \Gamma \cap \widetilde{\mathcal{H}_\tau}$.

Although we could obtain the finite presentation and VFL properties of $\Lambda^s_{g,r}$ using an equivariant spine for the action of Γ_τ on $\mathcal{T}_\tau$, we need a manifold bordification of $\mathcal{T}_\tau$ with compact quotient under the action of Λ in order to apply Theorem 1.1(b). (Harer constructs such a manifold as a neighborhood of the spine for Teichmüller space when F is orientable and $r + s > 0$, but we need Harvey's bordification for the case $r = s = 0$ anyway, so will not make use of Harer's construction.) To this end, we now recall some facts from [15, Chapter 3, §2]. For ε sufficiently small so that for any hyperbolic structure on $\widetilde{N}$, all simple closed geodesics of length $\leq \varepsilon$ are disjoint, define W_ε to be the subspace of $\mathcal{T}^{2s}_{\check{g},0}$ consisting of the hyperbolic structures in which every essential simple closed curve has length $\geq \varepsilon$. It is a codimension-zero smooth submanifold with corners of $\mathcal{T}^{2s}_{\check{g},0}$, invariant under the action of Γ, and W_ε/Γ is compact. There is an $\mathcal{H}$-equivariant (although stated in [15] only for $\Gamma^{2s}_{\check{g},0}$) flow that provides a deformation of $\mathcal{T}^{2s}_{\check{g},0}$ onto $W_\varepsilon - \partial W_\varepsilon$, and W_ε can be identified with Harvey's [16], [17] bordification of $\mathcal{T}^{2s}_{\check{g},0}$. Since the flow is equivariant, it deforms $\mathcal{T}_\tau$ onto $\mathcal{T}_\tau \cap (W_\varepsilon - \partial W_\varepsilon)$ which is therefore contractible.

Lemma 2.3.1. *There is a subgroup Λ of finite index Γ_τ such that $(\mathcal{T}_\tau \cap W_\varepsilon)/\Lambda$ is a compact aspherical manifold.*

Proof. Let Γ_1 be a torsion-free subgroup of finite index in Γ; replacing Γ_1 by $\Gamma_1 \cap \tau\Gamma_1\tau^{-1}$, we may assume that τ normalizes Γ_1. Let $\Lambda = \Gamma_1 \cap \Gamma_\tau$; then τ normalizes Λ as well. Therefore $\mathcal{T}_\tau \cap W_\varepsilon$ is invariant under Λ, so $(\mathcal{T}_\tau \cap W_\varepsilon)/\Lambda$ maps into the compact manifold W_ε/Γ_1. To deduce that $(\mathcal{T}_\tau \cap W_\varepsilon)/\Lambda$ is a compact manifold, we shall show that this map is an imbedding. Now Λ acts freely, properly discontinuously, and isometrically on $\mathcal{T}_\tau \cap W_\varepsilon$, so the quotient $(\mathcal{T}_\tau \cap W_\varepsilon)/\Lambda$ is a compact smooth

manifold with corners. Since τ normalizes Γ_1, it induces an isometric involution $\bar{\tau}$ on W_ε / Γ_1, whose fixed-point set must be a submanifold. But $\mathscr{T}_\tau \cap W_\varepsilon$ is the fixed-point set of a lift of $\bar{\tau}$ to W_ε, so the quotient of $\mathscr{T}_\tau \cap W_\varepsilon$ by its stabilizer Λ imbeds onto a component of the fixed-point set of $\bar{\tau}$. This completes the proof.

Since W_ε / Λ is a compact aspherical manifold whose fundamental group is isomorphic to a subgroup of finite index in Γ_τ, we have shown that $\Lambda^s_{g,0}$ (and hence, as remarked at the start of the proof of Theorem 2.1, $\Lambda^s_{g,r}$) is finitely presented and of type VFL.

To complete the proof of Theorem 2.1, we will show that the boundary of $\mathscr{T}_\tau \cap W_\varepsilon$ is homotopy equivalent to a wedge of spheres of the same dimension. For any 2-manifold F of finite type having (orientable or nonorientable) genus g, define a simplicial complex Y^s_g by taking as a k-simplex each isotopy class of $k+1$ pairwise disjoint simple closed curves $C_0, C_1, \cdots, C_k$ such that no component of $F - \bigcup_{i=0}^{k} C_i$ has nonnegative Euler characteristic. When F is orientable, Y^s_g is equal to the complex Z^s_g of [14]. When F is nonorientable, the dimension of Y^s_g is $2g+s-4$.

As explained in [17] and [12, Chapter 3, §2(iii)], the complex $Y^{2s}_{\tilde{g}}$ parametrizes the boundary of Harvey's bordification of $\mathscr{T}^{2s}_{\tilde{g},0}$. The points in the simplex $\langle [C_0, C_1, \cdots, C_k] \rangle$ parametrize the Teichmüller spaces of degenerate hyperbolic structures on $\tilde{N}$ in which the geodesics in the isotopy classes of $C_0, C_1, \cdots, C_k$ have been pinched to points. Consequently, ∂W_ε is equivariantly homotopy equivalent to $Y^{2s}_{\tilde{g}}$.

In our situation there is a natural way to imbed Y^s_g in $Y^{2s}_{\tilde{g}}$. Consider a $(k+l)$-simplex $\langle [A_0, A_1, \cdots, A_k, B_{k+1}, \cdots, B_{k+l}] \rangle$ in Y^s_g, with notation selected so that the A_i are 1-sided loops in N and the B_j are 2-sided. The preimage of each A_i in $\tilde{N}$ is a single loop A^1_i, while the preimage of each B_j consists of a pair $\{B^1_j, B^2_j\}$ of nonparallel loops interchanged by the covering transformation τ. Map the vertices of Y^s_g to the vertices of the first barycentric subdivision of $Y^{2s}_{\tilde{g}}$ by sending $\langle [A_i] \rangle$ to $\langle [A^1_i] \rangle$ and $\langle [B_j] \rangle$ to the barycenter of the 1-simplex $\langle [B^1_j, B^2_j] \rangle$. This extends to a simplicial imbedding of the first barycentric subdivision of Y^s_g into the first barycentric subdivision of $Y^{2s}_{\tilde{g}}$. The image of this imbedding is precisely the fixed-point set of the action of τ on $Y^{2s}_{\tilde{g}}$, and parametrizes the τ-invariant degenerations of τ-invariant hyperbolic structures on $\tilde{N}$. Thus Y^s_g is homotopy equivalent to the boundary of the bordification $\mathscr{T}_\tau \cap W_\varepsilon$ of the Teichmüller space of N.

Lemma 2.3.2. *There is a homotopy equivalence of pairs* $\Phi\colon (A_\infty^0, B_\infty^0)$ $\to ((Y_{\tilde{g}}^{2s})^0, (Y_g^s)^0)$, *whose restriction to* A_∞^0 *is* Γ-*equivariant and whose restriction to* B_∞^0 *is* Γ_τ-*equivariant.*

Proof. Follow the proof of Theorem 3.4 of [14]. The homotopy equivalence $\Psi\colon A_\infty^0 \to (Y_{\tilde{g}}^{2s})^0$ defined there carries B_∞^0 to $(Y_g^s)^0$. A cover of B_∞^0 is constructed as in the proof of Theorem 3.4. To see that the sets in this cover are contractible requires the contractibility of the complexes $B(\Delta)$ for nonorientable surfaces, which we have in Corollary 2.2.4. The constructions in the remainder of the argument are equivariant, so they show that the restriction of Ψ to B_∞^0 is a Γ_τ-equivariant homotopy equivalence.

Lemma 2.3.3. *Let* χ *be the Euler characteristic of a surface of (orientable or nonorientable) genus* g *with* s *punctures.*

(a) *If* $g = 0$, *then* Y_g^s *is homotopy equivalent to a wedge of spheres of dimension* $-\chi - 2$.

(b) *If the surface has genus at least* 1, *then* Y_g^0 *is homotopy equivalent to a wedge of spheres of dimension* $-\chi$, *while for* $s > 0$, Y_g^s *is homotopy equivalent to a wedge of spheres of dimension* $-\chi - 1$.

Proof. When the surface is orientable, this is Theorem 3.5 of [14]. Suppose that it is a nonorientable surface N. We first show that Y_g^s is $(g + s - 4)$-connected. Construct a τ-invariant subdivision A' of A by adding in the barycenter of each 1-simplex whose endpoints are a τ-invariant (up to isotopy) pair of arcs, and extending this subdivision to all of A in the obvious way. The fixed-point set of τ is a subcomplex T' of A'. Let $T'_\infty = T' \cap A_\infty$. By Corollary 2.2.4, T' is contractible. Assuming that $s > 0$, observe that fewer than $g + s - 1$ arcs cannot fill up N, hence the $(g + s - 3)$-skeleton of T' is contained in T'_∞. Since T' is contractible, it follows that T'_∞ and hence Y_g^s are $(g + s - 4)$-connected.

We are now in a position to adapt the proof of Theorem 3.5 of [14]. Assuming that $s > 0$ so that there are distinguished points $p_1, \cdots, p_s$, define $\widehat{Y}_q^s$ to be the subcomplex of Y_g^s consisting of those simplices $\langle [C_0, \cdots, C_k] \rangle$ for which no C_i bounds a disc which contains p_1 and one other p_i, and no C_i bounds a Möbius band which contains p_1 and no other p_i. The proof of Lemma 3.6 of [14] goes through to show that the map $\widehat{Y}_g^s \to Y_g^{s-1}$ defined by forgetting that p_1 is distinguished is a homotopy equivalence.

It remains to perform the inductive step. For $g = 1$, Y_1^0 and Y_1^1 are empty, and for $s > 1$, Y_1^s is $(s - 2)$-dimensional and $(s - 3)$-connected, yielding the values in Lemma 2.3.3(a) (using the convention that sets of

negative dimension are empty). So we assume that $g > 1$. We first argue for $s = 0$. For $g = 2$, Y_2^0 consists of three points so is homotopy equivalent to a wedge of two 0-dimensional spheres. So assume that $g > 2$. We follow the argument on [14, p. 171], using its notation. For the inductive step there, suppose that X_{k+1} has been shown to be homotopy equivalent to a complex of dimension no more than $-\chi$. Consider a vertex $\langle [C_0, \cdots, C_k] \rangle$ of $X_k - X_{k+1}$. Let $N^1, \cdots, N^t$ be the components of the surface obtained by splitting N along $C, \cdots, C_k$. Some of the N^i may be orientable. If χ_i equals the Euler characteristic of N^i, then $\chi = \Sigma_{i=1}^t \chi_i$. By induction, each $Y(N^i)$ is homotopy equivalent to a complex of dimension at most $-\chi_i - 1$, therefore the dimension of the link $\langle [C_0, \cdots, C_k] \rangle$ is at most

$$\Sigma_{i=1}^t (-\chi_i - 1) + (t - 1) = -\chi - 1.$$

The rest of the argument in [14] applies *mutatis mutandis*. This completes the proof of Lemma 2.3.3.

To complete the proof of Theorem 2.1, it is only necessary to apply Theorem 1.1(b) to the manifold $\mathcal{T}_\tau \cap W_\varepsilon$, using the fact that $\mathcal{T}_\tau \cap W_\varepsilon$ is $(3g - 6 + 2s)$-dimensional together with the values given in Lemma 2.3.3. This completes the proof of Theorem 2.1.

3. Mapping class groups of fibered 3-manifolds

Fibered 3-manifolds are I-bundles or Seifert fibered spaces. The two main results of this section are the following.

Corollary 3.2.2. *Let $(M, \underline{m})$ be an irreducible I-bundle over $(F, \underline{f})$. Then $\mathcal{H}(M, \underline{m})$ is a finitely presented virtual duality group of type VFL.*

Theorem 3.6.1. *Let $(M, \underline{m})$ be a compact orientable irreducible sufficiently large 3-manifold that is a Seifert fiber space with complete and useful boundary pattern. Let $\underline{m}_1 \subseteq \underline{m}$. Then $\mathcal{H}(M, \underline{m} \text{ rel } |\underline{m}_1|)$ is a finitely presented virtual duality group of type VFL.*

The double-underlined symbols indicate *boundary patterns*, a device introduced by Johannson. We will discuss these in §3.1. In §3.2, Corollary 3.2.2. is proved by identifying $\mathcal{H}_+(M, \underline{m})$ with a mapping class group of F. For the Seifert fibered case, which occupies all remaining sections, the *Dehn twists* about tori and annuli play a crucial role; these are introduced in §3.3. §3.4 contains results which imply Theorem 3.6.1 for several exceptional cases, such as the Seifert fibered manifolds which admit homeomorphisms which are not isotopic to fiber-preserving homeomorphisms.

In §3.5, we prepare for the general case by reviewing and extending work of Johannson on mapping class groups of Seifert 3-manifolds. This section also contains some technical results which will be needed in §4. The last section contains the proof of Theorem 3.6.1 and a variant needed in §4.

3.1. Boundary patterns. The following definitions are due to Johannson [21]. A *boundary pattern* $\underline{m}$ for a compact n-manifold M is a finite set of compact, connected $(n-1)$-manifolds in ∂M, such that the intersection of any i of them is empty or consists of $(n-i)$-manifolds. The symbol $|\underline{m}|$ will mean the set of points of ∂M that lie in some element of $\underline{m}$. A boundary pattern is said to be *complete* when $|\underline{m}| = \partial M$. Any boundary pattern $\underline{m}$ can be enlarged to a complete boundary pattern by adding in the closures of the components of the complement of $|\underline{m}|$.

The boundary pattern which consists of the components of the boundary of M is denoted by $\underline{\partial M}$.

If S is a codimension-zero submanifold of M, then a boundary pattern $\underline{m}$ for M induces a boundary pattern $\underline{s}$ for S defined by

$$\underline{s} = \{\text{components of } \mathrm{Fr}(S)\} \cup \bigcup_{F \in \underline{m}} \{\text{components of } F \cap S\},$$

where $\mathrm{Fr}(S)$ denotes the frontier of S in M. If $\underline{m}$ is complete, then so is $\underline{s}$. The boundary pattern consisting of the components of $\mathrm{Fr}(S)$ is denoted by $\underline{\mathrm{Fr}(S)}$.

An *admissible map* f from $(M, \underline{m})$ to $(N, \underline{n})$ is a map such that

$$\underline{m} = \bigcup_{G \in \underline{n}} \left\{\text{components of } f^{-1}(G)\right\}.$$

The notation $f: (M, \underline{m}) \to (N, \underline{n})$ indicates that f is admissible.

For an I-bundle $p: M \to F$ of manifolds with boundary pattern $(M, \underline{m})$ and $(F, \underline{f})$, it is always assumed that the boundary pattern $\underline{m}$ consists of $\{p^{-1}(B) | B \in \underline{f}\}$ together with the components of the closure of $\partial M - p^{-1}(\partial F)$. (Thus p is not an admissible map; indeed, it does not even carry boundary to boundary.) For a Seifert fibered 3-manifold $(\Sigma, \underline{\sigma})$ with boundary pattern, it is always assumed that the elements of $\underline{\sigma}$ are the preimages of the components of a boundary pattern of the orbit surface (so that the quotient map is admissible). Consequently the elements of $\underline{\sigma}$ must be tori or saturated annuli.

The *mapping class group* of a manifold with boundary pattern $(M, \underline{\underline{m}})$ is the set of path components of the topological group

$$\left\{ h \mid h \text{ is a homeomorphism of } M \text{ and } h(F) = F \text{ for every } F \in \underline{\underline{m}} \right\}.$$

We denote the mapping class group by $\mathscr{H}(M, \underline{\underline{m}})$.

Warning. Our definition is nonstandard in that we allow only homeomorphisms that carry each component of the boundary pattern to itself. Since we are concerned here only with virtual properties of the mapping class group, this is a harmless technical convenience.

An *i-faced 2-disc* is a 2-disc whose boundary pattern is complete and has i components. A boundary pattern for a 3-manifold is called *useful* when the boundary of every admissibly imbedded i-faced disc in $(M, \underline{\underline{m}})$ with $i \leq 3$ bounds a disc D in ∂M such that $D \cap \bigcup_{F \in \underline{\underline{m}}} \partial F$ is the cone on $\partial D \cap \bigcup_{F \in \underline{\underline{m}}} \partial F$. When the boundary pattern is the set of boundary components of M, this is equivalent to incompressibility of ∂M. In general, it implies that each component of the boundary pattern is incompressible in M, but says more. For example, the product of a 4-faced disc with S^1 yields a useful boundary pattern on the solid torus, but the product of a 3-faced disc with S^1 does not.

We will now restate the theorems of §2 for 2-manifolds with boundary pattern.

Lemma 3.1.1. *Let F be a 2-manifold of finite type with compact boundary and let $\underline{\underline{f}}$ be a boundary pattern for F. Let $\underline{f_1}$ be a (possibly empty) subset of $\underline{\underline{f}}$. Then $\mathscr{H}(F, \underline{\underline{f}} \text{ rel } |f_1|)$ is a finitely presented virtual duality group of type VFL. Let r be the number of components of ∂F that contain either an element of $\underline{\underline{f}}$ that is an arc, or an element of $\underline{f_1}$, and let s be the sum of the number that do not and the number of punctures of F. If F is orientable of genus g and $2g + r + s > 2$, then the dimension of $\mathscr{H}(F, \underline{\underline{f}} \text{ rel } |f_1|)$ is $d(g, r, s)$, where $d(g, 0, 0) = 4g - 5$, $d(g, r, s) = 4g + 2r + s - 4$ if $g > 0$ and $r + s > 0$, and $d(0, r, s) = 2r + s - 3$. If F is nonorientable of genus n and $n + r + s > 2$, then the dimension of $\mathscr{H}(F, \underline{\underline{f}} \text{ rel } |f_1|)$ is $d(n, r, s)$, where $d(n, 0, 0) = 2n - 5$ and $d(n, r, s) = 2n + 2r + s - 4$ if $r + s > 0$.*

Proof. Since $\mathscr{H}_+(F, \underline{\underline{f}} \text{ rel } |f_1|)$ contains a subgroup of finite index which is isomorphic to $\Gamma^s_{g,r}$ or $\Lambda^s_{n,r}$, the lemma follows from the theorems of §2.

3.2. Mapping class groups of I-bundles.

Proposition 3.2.1. *Let $(M, \underline{\underline{m}})$ be an irreducible I-bundle over $(F, \underline{f})$. Suppose that $\underline{\underline{m_1}}$ is a (possibly empty) subset of $\underline{\underline{m}}$ such that each element*

of m_1 is contained in the preimage of ∂F. Let $\underline{f_1}$ be the images of $\underline{m_1}$ in $\overline{\overline{F}}$. Then $\mathcal{H}_+(M, \underline{m} \text{ rel } |\underline{m}|) \cong \mathcal{H}_+(F, \underline{f} \text{ rel } |\underline{\underline{f_1}}|)$.

Proof. Assume for now that m_1 is empty. Suppose first that M is a product I-bundle. Define $\mathcal{H}_+(\overline{F}, \underline{f}) \to \mathcal{H}_+(M, \underline{m})$ by sending $\langle g \rangle$ to $\langle g \times 1_I \rangle$. This is injective because restriction to $F \times \{0\}$ defines a left inverse (the restriction is defined because in order to preserve the boundary pattern, a homeomorphism must preserve $F \times \{0\}$). By [21, Corollary 5.9], it is surjective.

Suppose that M is a twisted I-bundle over the nonorientable surface F. For $\langle g \rangle \in \mathcal{H}(F, \underline{f})$ there is a unique orientation-preserving lift $\tilde{g}$ of g to a homeomorphism of the orientable double cover $\widetilde{F}$. Since M is the mapping cylinder of the projection from $\widetilde{F}$ to F, and $\tilde{g}$ commutes with the covering transformation, the homeomorphism $\tilde{g} \times 1_I$ of $\widetilde{F} \times I$ induces an admissible homeomorphism of M. This defines a homeomorphism $\mathcal{H}_+(F, \underline{f}) = \mathcal{H}(F, \underline{f}) \to \mathcal{H}_+(M, \underline{m})$ which is surjective by [21, Corollary 5.9]. For injectivity, suppose first that F is a Klein bottle, then $\mathcal{H}(F) \cong \text{Out}(\pi_1(F))$ and it follows that the homomorphism is injective. In all other cases, the lifting homomorphism $\mathcal{H}(F, \underline{f}) \to \mathcal{H}(\widetilde{F}, \underline{\tilde{f}})$ is injective, so the homomorphism to $\mathcal{H}_+(M, \underline{m})$ is injective. This completes the proof when m_1 is empty. If it is nonempty, let $\underline{f_2}$ be the boundary pattern obtained from $\underline{f}$ by subdividing each element of $\underline{f_1}$ into three arcs. Let $\underline{m_2}$ be the corresponding boundary pattern of M. It is easy to see that $\overline{\overline{\mathcal{H}}}(M, \underline{m} \text{ rel } |\underline{m_1}|) \cong \mathcal{H}(M, \underline{m_2})$ and $\mathcal{H}(F, \underline{f} \text{ rel } |\underline{f_1}|) \cong \mathcal{H}(F, \underline{f_2})$, so the result now follows from the case when m_1 is empty.

From Lemma 3.1.1 and Proposition 3.2.1, we have immediately

Corollary 3.2.2. *Let $(M, \underline{m})$ be an irreducible I-bundle over $(F, \underline{f})$. Then $\mathcal{H}(M, \underline{m})$ is a finitely presented virtual duality group of type VF$\overline{L}$.*

3.3. Dehn twists. The mapping class groups of Seifert fibered 3-manifolds contain *Dehn twists* about tori, defined as follows. Regard S^1 as the unit circle in the complex plane. For $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ define a homeomorphism $t_{p,q}$ on $S^1 \times S^1 \times I$ by

$$t_{p,q}(\exp(\theta), \exp(\psi), s) = (\exp(\theta + 2\pi p s), \exp(\psi + 2\pi q s), s).$$

In general, if f is a homeomorphism of $S^1 \times S^1 \times I$ that restricts to the identity on $S^1 \times S^1 \times \partial I$, the *trace* of f is the element of $\pi_1(S^1 \times S^1, (1, 1))$ represented by the path which sends $s \in I$ to the

projection of $f(1, 1, s)$ to $S^1 \times S^1$. This agrees with the standard definition of trace when f is regarded as a homotopy of $S^1 \times S^1$. Combining Lemma 3.5 of [39] and the results of [12] shows that sending $\langle h \rangle \in \mathscr{H}(S^1 \times S^1 \times I \text{ rel } S^1 \times S^1 \times \partial I)$ to the trace of h yields an isomorphism from $\mathscr{H}(S^1 \times S^1 \times I \text{ rel } S^1 \times S^1 \times \partial I)$ to $\pi_1(S^1 \times S^1, (1, 1))$; thus composition of these homeomorphisms corresponds to addition of their traces. Moreover, this isomorphism shows that each isotopy class contains exactly one of the $t_{p,q}$.

Now suppose M contains an imbedded 2-torus $T = S^1 \times S^1$ either disjoint from or contained in ∂M. One can construct a homeomorphism of M using $t_{p,q}$ on a product region $T \times [0, 1] \subseteq M$ and the identity homeomorphism on the rest of M; this is called a *Dehn twist about* T. By uniqueness of product regions, the isotopy class of the Dehn twist depends only on the trace and not on the choice of product region.

If $S^1 \times S^1 \times I$ is Seifert fibered by the circles $S^1 \times \{t\} \times \{s\}$, then each $t_{p,q}$ takes fibers to fibers. When T is a fibered torus, we can choose the coordinates on the product region so that the fibers are these circles, and consequently the Dehn twists can and always will be chosen to be fiber-preserving. Notice that the Dehn twist takes each fiber to itself precisely when the trace is a multiple of the fiber. In this case the Dehn twist is said to be *vertical*.

Dehn twists about properly imbedded annuli are defined similarly. In the Seifert fibered case, Dehn twists about vertical annuli are always vertical.

3.4. Exceptional cases. There are several cases for which the proof of Theorem 3.6.1 requires special arguments. To avoid later distraction, we will deal with these exceptional cases in the five propositions in this subsection. In the process, we will determine the virtual cohomological dimension of their mapping class groups; this will be used in §9.

Proposition 3.4.1. *Let* $(M, \underline{m})$ *be an orientable Seifert 3-manifold which is an* S^1*-bundle over the annulus. Then* $\mathscr{H}(M, \underline{m})$ *is a finitely presented virtual duality group of type VFL, of dimension 2 if both components of* ∂M *contain an element of* $\underline{m}$ *which is an annulus, and of dimension 1 otherwise.*

Proof. Note that M is homeomorphic to $S^1 \times S^1 \times I$. If the boundary pattern is empty or consists of some components of ∂M, then $\mathscr{H}(M, \underline{m})$ contains

$$\mathscr{H}\left(S^1 \times S^1 \times I, S^1 \times S^1 \times \{0\}\right) \cong \mathscr{H}\left(S^1 \times S^1 \times \{0\}\right) \cong \mathrm{GL}(2, \mathbb{Z})$$

as a subgroup of index 2 or 1 (where the first isomorphism uses [39, Lemma 3.5]). The result follows in this case using Example 1.2(d).

Next, suppose that exactly one component of ∂M, say $S^1 \times S^1 \times \{0\}$, contains an annulus of $\underline{m}$. The Dehn twists about vertical annuli running from $S^1 \times S^1 \times \{0\}$ to $S^1 \times S^1 \times \{1\}$ generate an infinite cyclic subgroup of $\mathcal{H}(M, \underline{m})$. Now $S^1 \times S^1 \times \{0\}$ is a union of annuli that are either elements of $\underline{m}$ or the closures of complementary regions of $|\underline{m}|$. An (isotopy class of) orientation-preserving homeomorphism of $S^1 \times S^1 \times \{0\}$ that preserves each of these annuli must be one of the following: (1) a Dehn twist about the boundary of one of these annuli, and hence the restriction of a Dehn twist about a vertical annulus, or (2) if $\underline{m}$ consists of exactly one annulus, an involution which has two fixed points on the annulus, or (3) if $\underline{m}$ consists of exactly two annuli, an involution which has exactly two fixed points on each annulus. A homeomorphism of $S^1 \times S^1 \times I$ which is the identity on $S^1 \times S^1 \times \{0\}$ is isotopic to the identity relative to $S^1 \times S^1 \times \{0\}$ (by Lemma 3.5 of [39]). Therefore the infinite cyclic subgroup generated by a Dehn twist about a vertical annulus running from $S^1 \times S^1 \times \{0\}$ to $S^1 \times S^1 \times \{1\}$ has index at most 2 in $\mathcal{H}_+(M, \underline{m})$, showing that $\mathcal{H}(M, \underline{m})$ has dimension 1 in this case.

Finally, suppose that both boundary components of M contain annuli of $\underline{m}$. Then, a Dehn twist about $S^1 \times \{\frac{1}{2}\}$ is isotopic to the identity if and only if it is vertical. Further investigation shows that $\mathcal{H}_+(M, \underline{m})$ contains $\mathbb{Z} \times \mathbb{Z}$, generated by a Dehn twist about a vertical annulus and any nonvertical Dehn twist about $S^1 \times \{\frac{1}{2}\}$, as a subgroup of index at most 2. This completes the proof of Proposition 3.4.1.

Proposition 3.4.2. *Let $(M, \underline{m})$ be an orientable Seifert 3-manifold which is an S^1-bundle over the Möbius band. Then $\mathcal{H}(M, \underline{m})$ is finite.*

Proof. Observe that M is homeomorphic to the twisted I-bundle over the Klein bottle K. If the boundary pattern is empty or is $\{\partial M\}$, then from Proposition 3.2.1 we have $\mathcal{H}_+(M, \underline{m}) \cong \mathcal{H}_+(K) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Consider any other boundary pattern, which must consist of saturated annuli. A Dehn twist about ∂M whose trace is homotopic to a fiber is admissibly isotopic to the identity, and one of the Dehn twists about ∂M whose trace is a cross section to the fibering is isotopic to the identity (rel ∂M). Hence for these boundary patterns, the mapping class group is still finite. This completes the proof of Proposition 3.4.2.

Proposition 3.4.3. *Let $(M, \underline{m})$ be an orientable 3-manifold which is an S^1-bundle over the torus. Then $\mathcal{H}(M)$ is a finitely presented virtual*

duality group of type VFL, of dimension 3 *if the Euler class of* M *is zero, and of dimension* 1 *if it is nonzero.*

Proof. Since M is closed, irreducible, and sufficiently large, it is known from [39] that $\mathscr{H}(M) \cong \mathrm{Out}(\pi_1(M))$. If its Euler class is zero, then M is the 3-torus so $\mathrm{Out}(\pi_1(M)) \cong \mathrm{GL}(3, \mathbb{Z})$ and the result follows from Example 1.2(d). So suppose the Euler class n is nonzero. The fundamental group of M has presentation $\langle x, y, t | [x, t] = [y, t] = 1, [x, y] = t^n \rangle$. The center of $\pi_1(M)$ is infinite cyclic, generated by t, and there is a central extension

$$1 \to \mathbb{Z} \to \pi_1(M) \to \mathbb{Z} \times \mathbb{Z} \to 1.$$

Since the center is characteristic, we have a homomorphism

$$\alpha \colon \mathrm{Out}\left(\pi_1(M)\right) \to \mathrm{Out}(\mathbb{Z} \times \mathbb{Z}) \cong \mathrm{GL}(2, \mathbb{Z}).$$

There are (nonvertical) Dehn twists about vertical tori in M which induce the generators $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ of $\mathrm{SL}(2, \mathbb{Z})$, and there is a homeomorphism inducing $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (and reversing the orientation on the fiber). Therefore α is surjective. The elements of $\ker(\alpha)$ must induce the identity on the center as well, since $[x, y] = t^n$. Therefore an element of $\ker(\alpha)$ is determined by its effect on x and y, and must send x to some xt^i and y to some yt^j. Define commuting automorphisms β_1 and β_2 by $\beta_1(x) = xt$, $\beta_1(y) = y$, $\beta_1(t) = t$, $\beta_2(x) = x$, $\beta_2(y) = yt$, and $\beta_2(t) = t$. From the above, these generate $\ker(\alpha)$. Since conjugation by x equals β_2^n and conjugation by y equals β_1^{-n}, it follows that $\ker(\alpha) \cong \mathbb{Z}/n \times \mathbb{Z}/n$. It is well known that $\mathrm{GL}(2, \mathbb{Z})$ has a free subgroup of finite index (because $\mathrm{SL}(2, \mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ which is virtually free by [22 or 33, Lemma 7.4]). Therefore $\mathrm{Out}(\pi_1(M))$ contains a subgroup of finite index which is isomorphic to a semidirect product $(\mathbb{Z}/n \times \mathbb{Z}/n) \circ F$, where F is free, so $\mathrm{Out}(\pi_1(M))$ is virtually free. This completes the proof of Proposition 3.4.3.

Proposition 3.4.4. *Let* $(M, \underline{m})$ *be an orientable* 3-*manifold which is an* S^1-*bundle over the Klein bottle. If the Euler class of* M *is zero, then* $\mathscr{H}(M)$ *is a finitely presented virtual duality group of type VFL, of dimension* 1. *If the Euler class is nonzero, then* $\mathscr{H}(M)$ *is finite.*

Proof. Observe that $\pi_1(M)$ has presentation $\langle a, b, t | btb^{-1} = t^{-1}, ata^{-1} = t, bab^{-1} = a^{-1}t^n \rangle$, where n is the Euler class. Suppose first that $n = 0$. In this case we have an extension

$$1 \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \pi_1(M) \to \mathbb{Z}/2 \to 1$$

in which the kernel is the subgroup generated by b^2, a, and t, and the quotient is represented by b. The kernel is the unique maximal abelian

subgroup of finite index, so is characteristic. The action of the quotient $\mathbb{Z}/2$ on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is given by the matrix

$$\tau = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

One checks easily that the normalizer of τ in $\mathrm{GL}(3, \mathbb{Z})$ is

$$\mathrm{Norm}\,(\tau) = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & p & q \\ 0 & r & s \end{bmatrix} \middle| ps - qr = \pm 1 \right\} \cong \mathbb{Z}/2 \times \mathrm{GL}(2, \mathbb{Z}),$$

and that restriction from $\mathrm{Aut}(\pi_1(M))$ to $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \cong \mathrm{GL}(3, \mathbb{Z})$ is a homomorphism with image $\mathrm{Norm}(\tau)$. The image of $\mathrm{Inn}(\pi_1(M))$ is the subgroup generated by τ, so there is a surjective homomorphism from $\mathrm{Out}(\pi_1(M))$ to the quotient of $\mathrm{Norm}(\tau)$ by the subgroup generated by τ, which is isomorphic to $\mathbb{Z}/2 \times \mathrm{PGL}(2, \mathbb{Z})$. Similarly to Proposition 3.4.3, the kernel of $\mathrm{Out}(\pi_1(M)) \to \mathbb{Z}/2 \times \mathrm{PGL}(2, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ generated by the outer automorphisms that send b to ba and b to bt. Since $\mathrm{PGL}(2, \mathbb{Z})$ contains a free subgroup of finite index ($\mathrm{PSL}(2, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/3$), an argument similar to that in Proposition 3.4.3 shows that $\mathrm{Out}(\pi_1(M))$ is virtually free.

Suppose that $n \neq 0$. Examining the effect of conjugating an arbitrary element $a^i b^j t^k$ by a, b, and t shows that the subgroup generated by t is the unique maximal infinite cyclic normal subgroup, and hence is characteristic. (This also follows from standard results about Seifert manifolds, such as [30, Theorem 8.7].) The kernel of $\mathrm{Out}(\pi_1(M)) \to \mathrm{Out}(\langle a, b | bab^{-1} = a^{-1} \rangle) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by the commuting outer automorphisms represented by β_1 and β_2, where $\beta_1(a) = at$, $\beta_1(b) = b$, $\beta_1(t) = t$, $\beta_2(a) = a$, $\beta_2(b) = bt$, and $\beta_2(t) = t$. But β_1^{-2n} is conjugation by b^2 and β_2^{-2} is conjugation by t, and it follows that the kernel is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2n$. Therefore $\mathrm{Out}(\pi_1(M))$ is finite. This completes the proof of Proposition 3.4.4.

The remaining exceptional cases are the 3-manifolds that fiber over the 2-sphere with three exceptional orbits.

Lemma. *Let M be a Seifert fibered 3-manifold that fibers over the 2-sphere with three exceptional orbits. Then $\mathrm{Out}(\pi_1(M))$ is finite.*

Proof. The fundamental group $\pi_1(M)$ has presentation

$$\left\langle q_1, q_2, q_3, h | hq_i h^{-1} = q_i, a_i^{\alpha_i} = h^{\beta_i}, q_1 q_2 q_3 = h^b \right\rangle,$$

where (α_i, β_i) are the Seifert invariants, so $0 < \beta_i < \alpha_i$, but b can be

zero. For integers α_1, α_2, $\alpha_3 \geq 2$, let $Q(\alpha_1, \alpha_2, \alpha_3)$ denote the quotient

$$\pi_1(M) / \langle q_1, q_2, q_3 | q_1^{\alpha_1} = q_2^{\alpha_2} = q_3^{\alpha_3} = q_1 q_2 q_3 = 1 \rangle.$$

We claim that $\mathrm{Out}(Q(\alpha_1, \alpha_2, \alpha_3))$ is finite. We may assume that $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 \leq 1$ since otherwise $Q(\alpha_1, \alpha_2, \alpha_3)$ is finite [41, Theorem 4.7.1]. Then $Q(\alpha_1, \alpha_2, \alpha_3)$ is a planar discontinuous group, and using [41, Theorem 5.8.3 and Corollary 5.16.6], $\mathrm{Out}(Q(\alpha_1, \alpha_2, \alpha_3))$ can be identified with a subgroup of $\mathscr{H}(S^2, \{x_1, x_2, x_3\})$, where x_1, x_2, and x_3 are three distinct points in S^2. This group is finite.

To prove the lemma, we may assume that $\pi_1(M)$ is infinite, in which case the center C is the infinite cyclic subgroup generated by h. Let K_1 denote the kernel of the restriction $\mathrm{Out}(\pi_1(M)) \to \mathrm{Out}(C)$. From above, the kernel K of the homomorphism $K_1 \to \mathrm{Out}(\pi_1(M)/\langle h \rangle)$ has finite index in K_1. We will show K is trivial.

Suppose $\{\phi\} \in K$. After changing ϕ by an inner automorphism, we may assume that it induces the identity automorphism on Q, as well as on C, so that on the generators of $\pi_1(M)$, ϕ has the effect $\phi(h) = h$ and $\phi(q_i) = q_i h^{n_i}$ for some integers n_i. But $h^{\beta_i} = \phi(h^{\beta_i}) = \phi(q_i^{\alpha_i}) = q_i^{\alpha_i} h^{\alpha_i n_i} = h^{\beta_i} h^{\alpha_i n_i}$, hence all $n_i = 0$ and $\phi = 1_{\pi_1(M)}$. This completes the proof.

As an immediate consequence, we have

Proposition 3.4.5. *Let M be a Haken 3-manifold which is Seifert fibered over the 2-sphere with three exceptional orbits. Then $\mathscr{H}(M)$ is finite.*

3.5. Extensions of Johannson's results. Let $(M, \underline{\underline{m}})$ be a Seifert fibered 3-manifold with boundary pattern, and with orbit surface $(F, \underline{f})$. Denote by $\mathscr{H}_+^f(M, \underline{\underline{m}})$ the path components of the space of orientation-preserving homeomorphisms that take fibers to fibers and take each element of $\underline{\underline{m}}$ to itself. The next theorem refers to the *Hantsche-Wendt manifold*, which is a closed flat 3-manifold given by the Seifert invariants $\{-1; (n_2, 1); (2, 1), (2, 1)\}$ (see [30, pp. 133, 138; 6, pp. 478–481; 38, 13]).

Theorem 3.5.1. *Let $(M, \underline{\underline{m}})$ be a Seifert 3-manifold with boundary pattern. If $(M, \underline{\underline{m}})$ contains an incompressible 2-manifold which is a union of fibers, then the natural homomorphism $\mathscr{H}_+^f(M, \underline{\underline{m}}) \to \mathscr{H}_+(M, \underline{\underline{m}})$ is injective. If either*

(a) *some element of $\underline{\underline{m}}$ is an annulus, or*

(b) *M is not one of the exceptions 5.1.1 to 5.1.5 of [21], M is not an S^1-bundle over the annulus or Möbius band, M is not an S^1-bundle over*

the torus or Klein bottle which admits a cross section, and M is not the Hantsche-Wendt manifold,

then the natural homomorphism is surjective.

Theorem 3.5.1 is a compilation of known results. The proof of injectivity is sketched in [39, p. 85]. Surjectivity in case (a) is proved by the argument in [19, Lemma VI.19], and in case (b) it is proved in [38] (see also [30, Theorem 8.7]).

Let $\mathscr{H}^0(M, \underline{m})$ denote the subgroup of $\mathscr{H}_+^f(M, \underline{m})$ generated by homeomorphisms that take each fiber to itself. These are called the *vertical* mapping classes.

Notation. Let $\mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$ denote the path components of the space of homeomorphisms that take each exceptional point (image in F of an exceptional fiber of M) of F to an exceptional point that has the same Seifert invariants associated to it, take each element of $\underline{f}$ to itself, and restrict to the identity on $|\underline{f_1}|$. If $\mathscr{E}$ is the set of exceptional points, then $\mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$ is isomorphic to a subgroup of finite index in $\mathscr{H}_+(F - \mathscr{E}, \underline{f} \text{ rel } |\underline{f_1}|)$.

There is a natural homomorphism from $\mathscr{H}_+^f(M, \underline{m} \text{ rel } |\underline{m_1}|)$ to $\mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$. The argument given in Proposition 25.3 of [21] shows that it is surjective (see also the discussion in Lemma 3.5.7 below). Thus we have

Theorem 3.5.2. *Let $(M, \underline{m})$ be an orientable Seifert fiber space with complete boundary pattern. Let $\underline{m_1} \subseteq \underline{m}$ and let $\underline{f}$ and $\underline{f_1}$ be the images of $\underline{m}$ and $\underline{m_1}$, respectively, in the orbit surface F. Then there is a short exact sequence*

$$1 \to \mathscr{H}^0\left(M, \underline{m} \text{ rel } \left|\underline{m_1}\right|\right) \to \mathscr{H}_+^f\left(M, \underline{m} \text{ rel } \left|\underline{m_1}\right|\right)$$
$$\to \mathscr{H}^*\left(F, \underline{f} \text{ rel } \left|\underline{f_1}\right|\right) \to 1.$$

It is necessary to discuss the subgroup $\mathscr{H}^0(M, \underline{m} \text{ rel } |\underline{m_1}|)$ in detail. Suppose first that the quotient surface F is orientable. The boundary pattern $\underline{m_1}$ consists of some of the boundary tori, say $T_1, T_2, \cdots, T_k$, and some vertical annuli $A_1, A_2, \cdots, A_l$. Without loss of generality, we may assume that the annuli A_i are disjoint (for if they intersect, they can be combined into larger annuli or tori, forming a new boundary pattern for which $\mathscr{H}^0(M, \underline{m} \text{ rel } |\underline{m_1}|)$ is the same as before). Let $\underline{m_2} = \underline{m} - \underline{m_1}$. The T_i project to some of the boundary circles of F, and the A_j project

to disjoint arcs in the boundary of F. The collection of these circles and arcs is $\underline{\underline{f_1}}$; let $\underline{\underline{f_2}} = \underline{\underline{f}} - \underline{\underline{f_1}}$. Now choose a standard set of generators for $H_1(F, |\underline{\underline{f_2}}|)$ consisting of properly imbedded arcs parallel to the images of the A_j, together with the circles of $\underline{\underline{f_1}}$, together with a dual pair of simple closed curves for each handle of $\overline{\overline{F}}$, and with a collection of $d-1$ arcs each running between two boundary components which are not entirely contained in $|\underline{\underline{f_1}}|$ (where d is the total number of such boundary components). The only homology relation among these chosen generators is that the sum of the arcs and circles of $\underline{\underline{f_1}}$ (with suitably chosen orientations) is homologous into $|\underline{\underline{f_2}}|$. The preimages in M of this set of generators are a collection of vertical annuli and tori, and as is proven on pp. 191–193 of [21], the Dehn twists about them generate $\mathscr{H}^0(M, \underline{\underline{m}} \text{ rel } |\underline{\underline{m_1}}|)$. Moreover, Johannson shows that except in a few exceptional manifolds (the S^1-bundles over the torus or Klein bottle with nonzero Euler class), the isotopy relations among these vertical Dehn twists correspond to the homological relations among the corresponding homology generators of $H_1(F, |\underline{\underline{f_2}}|)$. The exceptional fibers have no effect here; a vertical Dehn twist about a torus bounding a neighborhood of an exceptional fiber is isotopic to the identity, taking each fiber to itself. So far we have explained the case of F orientable for the following lemma, which relativizes Lemma 25.2 of [21].

Lemma 3.5.3. *Let $(M, \underline{\underline{m}})$ be an orientable Seifert fiber space with complete boundary pattern, but not one of the exceptions 5.1.1–5.1.5 of [21]. Assume further that M is not an S^1-bundle over the torus or Klein bottle. Let $\underline{\underline{m_1}} \subseteq \underline{\underline{m}}$, let $\underline{\underline{m_2}}$ be the complement of $\underline{\underline{m_1}}$ in $\underline{\underline{m}}$, and let $\underline{\underline{f}}$, $\underline{\underline{f_1}}$, and $\underline{\underline{f_2}}$ be the images of $\underline{\underline{m}}$, $\underline{\underline{m_1}}$, and $\underline{\underline{m_2}}$, respectively, in the orbit surface F. Then $\mathscr{H}^0(M, \underline{\underline{m}} \text{ rel } |\underline{\underline{m_1}}|) \cong \mathscr{H}_1(F, |\underline{\underline{f_2}}|)$.*

We will now discuss Lemma 3.5.3 in the case when F is nonorientable. Again, we choose a standard kind of basis for $H_1(F, |\underline{\underline{f_2}}|)$, but instead of a pair of dual curves in the handles of F, we regard $\overline{\overline{F}}$ as having cross-caps and choose the one-sided circles in these crosscaps as the homology generators in place of the dual pairs of circles on the 1-handles that were used in the orientable case. There is again a homological relation: the sum of the arcs and boundary circles in $\underline{\underline{f_1}}$, and twice of all the chosen one-sided circles, is homologous into $\underline{\underline{f_2}}$. The preimage in M of each one-sided circle is a one-sided Klein bottle K in M. Let $N(K)$ be a

fibered neighborhood of K in M. In Lemma 25.1 of [21], Johannson constructs a vertical homeomorphism supported in $N(K)$; on K it is a Dehn twist about the unique (up to isotopy) two-sided nonseparating simple closed curve in K (which is a fiber of the Seifert fibering), and since the lift of this to the orientable double cover of K is isotopic to the identity, this Dehn twist extends to a vertical homeomorphism of $N(K)$ which is the identity on the boundary torus, so extends to M. Johannson's analysis of this homeomorphism shows that its square is isotopic to a vertical Dehn twist about the torus $\partial N(K)$, and thus the correspondence between isotopy of vertical mapping classes and homological equivalence of the elements of $H_1(F, |f_2|)$ extends to the case when F is nonorientable. This completes our explanation of Lemma 3.5.3. A detailed proof can be obtained by modifying pp. 188–195 of [21].

Definition. If F is orientable, define $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m}_1|)$ to be $\mathscr{H}^0(M, \underline{m} \text{ rel } |\underline{m}_1|)$. If F is nonorientable, define $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m}_1|)$ to be the subgroup of $\mathscr{H}^0(M, \underline{m} \text{ rel } |\underline{m}_1|)$ generated by the squares of all Dehn twists about vertical tori and annuli, together with all Dehn twists about all vertical tori and annuli *whose images in F have even (transverse) intersection number with every one-sided circle in F*.

The reason for introducing $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m}_1|)$ is the following. In §4, we will need a free abelian subgroup of finite index in $\mathscr{H}^0(M, \underline{m} \text{ rel } |\underline{m}_1|)$ invariant under the action of $\mathscr{H}^*(F, \underline{f}, \text{ rel } |\underline{f}_1|)$ in the exact sequence of Theorem 3.5.2, which contains the subgroup generated by the vertical Dehn twists about the boundary circles and arcs in $\underline{f}_1$ *as a direct summand*.

We will now present several lemmas needed in the remainder of §3, and in §4. Suppose that F is nonorientable with r crosscaps, and let h_i denote the vertical homeomorphism defined as above, supported in a neighborhood of the preimage K_i of the one-sided circle in the i th crosscap. Observe that on any vertical Klein bottle, each element of $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m}_1|)$ restricts to a homeomorphism isotopic to the identity, hence none of the h_i's is in $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m}_1|)$. Define a homomorphism ϕ from $H_1(F, |f_2|)$ to $(\mathbb{Z}/2)^r \times (\mathbb{Z}/2)^d$ whose i th coordinate function, for $i \leq r$, is given by intersection number with the one-sided circle in the i th crosscap, and whose $(r+j)$ th coordinate function is given by intersection number with the j th of the boundary circles which are not

entirely contained in $|f_1|$. The kernel of ϕ is free abelian, generated by the arcs and boundary circles in f_1, together with the boundary circles of the crosscaps, together with two times each of the $d-1$ arcs connecting those boundary components, subject to the relation that the sum of the arcs and boundary circles of f_1 and the boundary circles of the crosscaps is homologous into $|f_2|$. We have

Lemma 3.5.4. *Let F be nonorientable. Under the identification of $\mathcal{H}^0(M, \underline{m}$ rel $|m_1|)$ with $H_1(F, |f_2|)$, $\mathcal{H}_T^0(M, \underline{m}$ rel $|m_1|)$ corresponds to the kernel of ϕ.*

Proof. Any element in the kernel of ϕ is $(\mathbb{Z}/2)$-homologous to a sum of arcs in ∂F and closed loops in the complement of the chosen crosscaps, so has even intersection number with every one-sided circle in F. It follows that each element of the kernel corresponds to an element of $\mathcal{H}_T^0(M, \underline{m}$ rel $|m_1|)$. On the other hand, the generators of $\mathcal{H}_T^0(M, \underline{m}$ rel $|m_1|)$ are of two types: (1) a Dehn twist about the preimage of a circle or arc representing an element of the kernel of ϕ (note that an arc running between two different boundary components intersects some one-sided circles in one point), or (2) the square of some vertical Dehn twist, which corresponds to an element in homology which is divisible by 2, and therefore is in the kernel of ϕ. This completes the proof.

Lemma 3.5.5. $\mathcal{H}_T^0(M, \underline{m}$ rel $|m_1|)$ *is normal in* $\mathcal{H}_+^f(M, \underline{m}$ rel $|m_1|)$.

Proof. Suppose v is a vertical Dehn twist about the preimage of a circle or arc α in F. Suppose $\langle g \rangle \in \mathcal{H}_+^f(M, \underline{m}$ rel $|m_1|)$, and g induces $\overline{g}$ on F. Then gvg^{-1} is a vertical Dehn twist about the preimage of $\overline{g}(\alpha)$. If α intersects each one-sided circle in F an even number of times, then the same is true for $\overline{g}(\alpha)$.

The next lemma, important in §4, follows from Lemmas 3.5.3 and 3.5.4.

Lemma 3.5.6. $\mathcal{H}_T^0(M, \underline{m}$ rel $|m_1|)$ *is a free abelian group, and the subgroup generated by the vertical Dehn twists about $T_1, T_2, \cdots, T_k$, $A_1, \cdots, A_l$ is a direct summand. If F is orientable, then the sum of these Dehn twists is trivial (assuming the traces have been chosen to be equal to the same orientation of the fiber) and all relations among them are multiples of this relation. If F is nonorientable, then they are free generators of this summand.*

We must also examine the homomorphism $\mathcal{H}_+^f(M, \underline{m}$ rel $|m_1|) \rightarrow \mathcal{H}^*(F, \underline{f}$ rel $|f_1|)$.

Lemma 3.5.7.   *If either F has nonempty boundary, or the exceptional set is nonempty, then the homomorphism $\mathscr{H}_+^f(M, \underline{m} \text{ rel } |\underline{m_1}|) \rightarrow \mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$ admits a splitting.*

Proof. This is observed in [21], but we will construct it explicitly for later reference. Let $\mathscr{E}$ denote as usual the set of exceptional points of F. By hypothesis, $F - \mathscr{E}$ is not a closed surface, therefore the circle bundle over $F - \mathscr{E}$ obtained by removing the exceptional fibers from M admits a cross section s. Given a homeomorphism f representing an element in $\mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$, consider the homeomorphism sfs^{-1} defined on $s(F-\mathscr{E})$. Since M is orientable, sfs^{-1} preserves the action of elements of $\pi_1(F-\mathscr{E})$ on the fibers, and since f represents an element of $\mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$, sfs^{-1} takes points near an exceptional fiber to points near an exceptional fiber having the same Seifert invariants. Therefore sfs^{-1} extends to an orientation-preserving homeomorphism of M.

Summarizing, we have:

Proposition 3.5.8.   *Let $(M, \underline{m})$ be an orientable Seifert fiber space with complete boundary pattern, but not one of the exceptions 5.1.1–5.1.5 of [21]. Assume further that M is not an S^1-bundle over the torus or Klein bottle. Let $\underline{m_1} \subseteq \underline{m}$. If the quotient surface F either has nonempty boundary or has nonempty exceptional set, then*

(a) $\mathscr{H}_+^f(M, \underline{m} \text{ rel } |\underline{m_1}|) \cong H_1(F, |\underline{f_2}|) \circ \mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$.

(b) $\mathscr{H}_+^f(M, \underline{m} \text{ rel } |\underline{m_1}|)$ *contains* $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m_1}|) \circ \mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$ *as a subgroup of finite index. The group $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m_1}|)$ is free abelian and the vertical Dehn twists about $T_1, T_2, \cdots, T_k, A_1, \cdots, A_l$ generate a direct summand of $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m_1}|)$.*

(c) *In both of these semidirect products, the action of $\mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$ on the normal subgroup agrees with the natural action of $\mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$ on $H_1(F, |\underline{f_2}|)$.*

Proof. By Theorem 3.5.2 and Lemma 3.5.7, $\mathscr{H}_+^f(M, \underline{m} \text{ rel } |\underline{m_1}|) \cong H^0(M, \underline{m} \text{ rel } |\underline{m_1}|) \circ \mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$, so part (a) follows from Lemma 3.5.3. Since $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m_1}|)$ is normal, $\mathscr{H}_T^0(M, \underline{m} \text{ rel } |\underline{m_1}|) \circ \mathscr{H}^*(F, \underline{f} \text{ rel } |\underline{f_1}|)$ is a subgroup, and so (b) follows from Lemmas 3.5.4 and 3.5.6. In the correspondence of Lemma 3.5.3, homology generators

of $H_1(F, |\underline{\underline{f_2}}|)$ that are 2-sided circles or arcs correspond to the vertical Dehn twists about their preimage tori or annuli, while the 1-sided circles correspond to the vertical homeomorphisms supported near their preimage Klein bottles, as described after Lemma 3.5.3. Suppose T is a vertical torus or annulus, and v is a vertical Dehn twist about T. If h is a fiber-preserving homeomorphism of M, then hvh^{-1} is a vertical Dehn twist about $h(T)$. The vertical homeomorphisms near Klein bottles behave in a similar manner. Thus the conjugation action of h corresponds to the action of h on $H_1(F, |\underline{\underline{f_2}}|)$, which is statement (c).

We end this section with one last lemma needed for Theorem 3.6.1.

Lemma 3.5.9. *Let* $(M, \underline{\underline{m}})$ *be an orientable Seifert fiber space with complete boundary pattern, but not one of the exceptions 5.1.1–5.1.5 of* [21]. *Assume further that* M *is not an* S^1-*bundle over the torus or Klein bottle. Let* $\underline{\underline{m_1}} \subseteq \underline{\underline{m}}$. *Then there is a subgroup of finite index in* $\mathscr{H}(M, \underline{\underline{m}} \text{ rel } |\underline{\underline{m_1}}|)$ *that intersects* $\mathscr{H}^0(M, \underline{\underline{m}} \text{ rel } |m_1|)$ *in a torsion-free subgroup.*

Proof. We will assume that M is closed and has no exceptional fibers, since otherwise the lemma follows directly from Proposition 3.5.8(b). If F is orientable, then $\mathscr{H}^0(M) \cong H_1(F)$ is already torsion-free, so assume that F is nonorientable. Choose a standard generating set $\{x_1, x_2, \cdots, x_g\}$ for $\pi_1(F)$, where each x_i is the center circle of a crosscap. The x_i satisfy the relation $x_1^2 x_2^2 \cdots x_g^2 = 1$. There is a presentation for $\pi_1(M)$ (see for example [19, p. 91])

$$\left\langle x_1, \cdots, x_g, h \mid x_i h x_i^{-1} = h^{-1} \text{ for } 1 \le i \le g, \ x_1^2 x_2^2 \cdots x_g^2 = h^b \right\rangle,$$

where b is the Euler class. Define G to be the finite abelian group

$$G = \left\langle X_1, \cdots, X_g, H \mid 2H = 0, \ 2X_k = 0 \text{ for } k \ge 2, \ 2X_1 = bH \right\rangle.$$

Obviously H is nonzero in G. Write π for $\pi_1(M)$. Now $\text{Aut}(\pi)$ acts on the right on $\text{Hom}(\pi, G)$; since G is abelian, the inner automorphisms act trivially and there is an induced action of $\text{Out}(\pi)$.

We have seen in the discussion of Lemma 3.5.3 that $\mathscr{H}^0(M)$ is generated by vertical Dehn twists about vertical annuli and tori, together with the homeomorphisms h_i, $1 \le i \le g$, such that the unique torsion element of $\mathscr{H}^0(M) \cong H_1(F)$ is represented by $\prod_{i=1}^{g} h_i$.

Let α be the element of $\text{Hom}(\pi, G)$ which sends the generators to their capitalizations. Now the induced automorphism $(h_i)_\#$ sends x_i to $x_i h$ while fixing all other generators, so the result of acting on α by the

outer automorphism induced by $\prod_{i=1}^{g} h_i$ is a homomorphism that sends each x_i to $X_i + H$. Let $\mathrm{Out}^0(\pi)$ denote the outer automorphisms that act trivially on $\mathrm{Hom}(\pi, G)$. Since $\mathrm{Hom}(\pi, G)$ is finite, this is a subgroup of finite index in $\mathrm{Out}(\pi)$, and therefore the subgroup of $\mathscr{H}(M)$ consisting of the mapping classes that induce automorphisms in $\mathrm{Out}^0(\pi)$ is a subgroup of finite index that does not contain the torsion element of $\mathscr{H}^0(M)$. This completes the proof of Lemma 3.5.9.

3.6. Statement and proof of Theorem 3.6.1. We are finally set up to prove the main result of §3.

Theorem 3.6.1. *Let $(M, \underline{m})$ be a compact irreducible sufficiently large 3-manifold that is a Seifert fiber space with complete and useful boundary pattern. Let $\underline{m_1} \subseteq \underline{m}$. Then $\mathscr{H}(M, \underline{m}$ rel $|\underline{m_1}|)$ is a finitely presented virtual duality group of type VFL.*

Proof. Since M is assumed to be sufficiently large, and its boundary pattern is useful, $(M, \underline{m})$ can only be among the exceptions 5.1.1–5.1.5 of [21] if it is Seifert fibered over the 2-sphere with three exceptional orbits. By Proposition 3.4.5, the mapping class group is finite for this case. For the S^1-bundles over the annulus, Möbius band, torus, or Klein bottle, Propositions 3.4.1–3.4.4 show that the mapping class groups are finitely presented virtual duality groups of type VFL. If M is the Hantsche-Wendt manifold, then $\mathscr{H}(M)$ is finite [6]. For the remainder of the proof, we will assume that M is not one of these cases, and hence by Theorem 3.5.1 that $\mathscr{H}_+(M, \underline{m}$ rel $|\underline{m_1}|) \cong \mathscr{H}_+^f(M, \underline{m}$ rel $|\underline{m_1}|)$.

According to Theorem 3.5.2, there is an exact sequence

$$1 \to \mathscr{H}^0\left(M, \underline{m} \text{ rel } \left|\underline{m_1}\right|\right) \to \mathscr{H}_+^f\left(M, \underline{m} \text{ rel } \left|\underline{m_1}\right|\right)$$
$$\to \mathscr{H}^*\left(F, \underline{f} \text{ rel } \left|\underline{f_1}\right|\right) \to 1.$$

Recall that $\mathscr{H}^*(F, \underline{f}$ rel $|\underline{f_1}|)$ is isomorphic to a subgroup of finite index in $\mathscr{H}(F - \mathscr{E}, \underline{f}$ rel $|\underline{f_1}|)$, where $\mathscr{E}$ denotes the set of exceptional points in F. By Lemma 3.1.1, $\mathscr{H}(F - \mathscr{E}, \underline{f}$ rel $|\underline{f_1}|)$ is a finitely presented virtual duality group of type VFL, hence so is $\mathscr{H}^*(F, \underline{f}$ rel $|\underline{f_1}|)$. Theorem 3.6.1 now follows using Lemma 3.5.9.

For use in §4 we need a variant of Theorem 3.6.1. Let $(\Sigma, \underline{\sigma})$ be a fibered 3-manifold and $\underline{\sigma_1} \subseteq \underline{\sigma}$. Define $\mathscr{G}(\Sigma, \underline{\sigma}; \underline{\sigma_1})$ to be the normal subgroup of $\mathscr{H}^f(\Sigma, \underline{\sigma})$ consisting of the mapping classes $\langle f \rangle$ such that the restriction of f to each element of $\underline{\sigma_1}$ is isotopic to the identity.

Lemma 3.6.2. *Let* $(\Sigma, \underline{\sigma})$ *be a fibered* 3-*manifold and* $\underline{\sigma}_1$ *a nonempty subset of* $\underline{\sigma}$. *If* $(\Sigma, \underline{\sigma})$ *is an I-bundle, assume that the elements of* $\underline{\sigma}_1$ *lie in the preimage of* ∂F. *Then* $\mathscr{G}(\Sigma, \underline{\sigma}; \underline{\sigma}_1)$ *is a finitely presented virtual duality group of type VFL.*

Proof. If $\partial\Sigma$ is empty, then this is a case of Theorem 3.6.1, so we assume that $\partial\Sigma$ is nonempty. Suppose first that $(\Sigma, \underline{\sigma})$ is an I-bundle. Since the elements of $\underline{\sigma}_1$ are annuli or squares, their mapping class groups are finite, so $\mathscr{G}(\Sigma, \underline{\sigma}; \underline{\sigma}_1)$ has finite index in $\mathscr{H}(\Sigma, \underline{\sigma})$ and the result follows from Corollary 3.2.2.

Assume now that Σ is Seifert fibered. According to Lemma 3.5.7, there is a splitting $\alpha \colon \mathscr{H}^*(F, \underline{f}) \to \mathscr{H}_+^f(\Sigma, \underline{\sigma})$ so that $\mathscr{H}_+^f(\Sigma, \underline{\sigma}) = \mathscr{H}^0(\Sigma, \underline{\sigma}) \circ \alpha(\mathscr{H}^*(F, \underline{f}))$. The splitting has the property that for $G \in \underline{\sigma}$, $\alpha(g)|_G$ is isotopic to the identity if and only if the restriction of g to the image of G in ∂F is orientation-preserving. Therefore the intersection of $\mathscr{G}(\Sigma, \underline{\sigma}; \underline{\sigma}_1)$ with $\alpha(\mathscr{H}^*(F, \underline{f}))$ has finite index in $\alpha(\mathscr{H}^*(F, \underline{f}))$. It follows that $\mathscr{G}(\Sigma, \underline{\sigma}; \underline{\sigma}_1)$ is an extension of $\mathscr{G}(\Sigma, \underline{\sigma}; \underline{\sigma}_1) \cap \mathscr{H}^0(\Sigma, \underline{\sigma})$ by a subgroup of finite index in $\mathscr{H}^*(F, \underline{f})$. Lemma 3.5.9 shows that there is a subgroup of finite index in $\mathscr{H}(\Sigma, \underline{\sigma})$ that intersects $\mathscr{G}(\Sigma, \underline{\sigma}; \underline{\sigma}_1) \cap \mathscr{H}^0(\Sigma, \underline{\sigma})$ in a torsion-free subgroup, and the result follows.

4. Mapping class groups of Haken manifolds

In this section we will prove one of our main results:

Theorem 4.3.1. *Let* $(M, \underline{m})$ *be a compact orientable sufficiently large* 3-*manifold with complete and useful boundary pattern. Then* $\mathscr{H}(M, \underline{m})$ *is a finitely presented virtual duality group of type VFL.*

In particular, since the boundary pattern $\underline{\partial M}$ is useful when the boundary of M is incompressible (see §3.1), this includes as a special case

Corollary 4.3.5. *Let* M *be a Haken* 3-*manifold. Then* $\mathscr{H}(M)$ *is a finitely presented virtual duality group of type VFL.*

The proof is long, and rather delicate in places. The first step is to determine that, apart from one exceptional case, the group $\mathscr{H}(M, \Sigma, \underline{m})$ of mapping classes which preserve Johannson's characteristic submanifold Σ is isomorphic to $\mathscr{H}(M, \underline{m})$. This is carried out, using a major result of Laudenbach, in §4.1. Also in that section, we handle the exceptional case of 3-manifolds admitting a Sol structure, by proving that their mapping class groups are finite.

The second part of the proof, detailed in §4.2, involves a subgroup $\mathscr{K}(M, \Sigma_1, \cdots, \Sigma_n, \underline{m})$ of $\mathscr{K}(M, \Sigma, \underline{m})$. This is the subgroup consisting of all mapping classes which preserve each component Σ_i of Σ, and whose restriction to the closure of the complement of Σ in M is admissibly isotopic to the identity. Results of Johannson show that $\mathscr{K}(M, \Sigma_1, \cdots, \Sigma_n, \underline{m})$ has finite index in $\mathscr{K}(M, \Sigma, \underline{m})$, so to prove Theorem 4.3.1 it suffices to prove that $\mathscr{K}(M, \Sigma_1, \cdots, \Sigma_n, \underline{m})$ is a finitely presented virtual duality group of type VFL.

Restriction to the components of Σ produces an exact sequence

$$ 1 \to K \to \mathscr{K}\left(M, \Sigma_1, \cdots, \Sigma_n, \underline{m}\right) \to \prod_{i=1}^{n} \mathscr{G}\left(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}\left(\Sigma_i\right)\right) \to 1, $$

where $\mathrm{Fr}(\Sigma_i)$ denotes the frontier of Σ_i in M, and the groups $\mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$ were defined just before Lemma 3.6.2. The kernel K consists of the mapping classes whose restriction to each Σ_i is isotopic to the identity. Suppose f is a homeomorphism representing an element of K. The restriction of f to the complement of Σ is isotopic to the identity, and its restriction to each Σ_i is isotopic to the identity. Therefore, f is isotopic preserving Σ to a homeomorphism which is the identity outside a product neighborhood of the frontier of Σ. That is, f is isotopic to a product of Dehn twists about the tori and annuli which make up the frontier of Σ. On the other hand, any such product is obviously an element of K, since after restriction to Σ_i, the components of the frontier are free to move so the Dehn twists about them are isotopic to the identity. These Dehn twists commute (see §3.3). Summarizing, we have the following:

Lemma 4.2.2. *There is a surjective homomorphism*

$$ \rho : \mathscr{K}\left(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m}\right) \to \prod_{i=1}^{n} \mathscr{G}\left(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i)\right) $$

whose kernel is the finitely generated abelian subgroup K generated by Dehn twists about the components of the frontier of Σ.

The surjectivity of ρ is clear: any element of $\prod_{i=1}^{n} \mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$ can be represented by a tuple $(g_1, g_2, \cdots, g_n)$ of homeomorphisms of Σ_i for which each g_i is the identity on the frontier of Σ_i (by definition of $\mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$), and taking each g_i on Σ_i and the identity on $M - \Sigma$ defines an element of $\mathscr{K}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ which restricts to the element we started with in $\prod_{i=1}^{n} \mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$.

By Lemma 3.6.2, $\prod_{i=1}^{n} \mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$ is a finitely presented virtual duality group of type VFL. If $\overline{K}$ were free abelian, then it would follow that $\mathscr{K}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ is a finitely presented virtual duality

group of type VFL. Surprisingly, K can contain torsion. A rather simple example is given at the end of §4.2. So to prove Theorem 4.3.1, it is necessary to find a subgroup of finite index in $\mathscr{H}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ which intersects K in a torsion-free subgroup. This is the content of §4.3 and is the most delicate part of the argument. A homomorphism is constructed from a subgroup of finite index in $\mathscr{H}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ to a finite group, which is injective on the torsion of K. Its kernel is the required subgroup.

4.1. The *Sol* exception. Recall that if M is a torus bundle over the circle such that the matrix of the attaching homeomorphism has trace of absolute value ≤ 2, then M is Seifert fibered, while if the trace has absolute value greater than 2, then M is not Seifert fibered but instead admits a geometric structure modeled on the geometry *Sol* (see for example [32, Theorem 5.5]).

Proposition 4.1.1. *Let $(M, \underline{m})$ be a Haken 3-manifold with complete and useful boundary pattern, and let Σ be its characteristic submanifold. Assume that M is not a torus bundle over the circle which admits a Sol structure. Then the natural homomorphism $\mathscr{H}(M, \Sigma, \underline{m}) \to \mathscr{H}(M, \underline{m})$ is an isomorphism.*

Proof. Since the characteristic submanifold is unique up to admissible isotopy, the homomorphism is surjective. In the proof of injectivity, we will eventually apply the following result of Laudenbach [24, pp. 50–62].

Theorem. *Let G be a compact connected incompressible surface in a $\mathbb{P}^2$-irreducible 3-manifold M and let m_0 be a basepoint in the interior of G. If $f: M \to M$ and $g: M \to M$ are homeomorphisms which preserve G, such that g is isotopic to f through homeomorphisms fixing m_0, and preserving ∂G, then the isotopy is deformable relative to $M \times \partial I \cup \partial M \times I$ to an isotopy through homeomorphisms that preserve G.*

Let $\langle f \rangle \in \mathscr{H}(M, \Sigma, \underline{m})$ and suppose that $H: M \times I \to M$ is an admissible isotopy from f to 1_M. To prove injectivity, we must find an isotopy that preserves $\mathrm{Fr}(\Sigma)$.

Let F be a component of $\mathrm{Fr}(\Sigma)$. It is easy to see that $f(F) = F$. We may assume that f fixes a basepoint m_0 in F. Recall that the *trace* of H at m_0 is the homotopy class in $\pi_1(M, m_0)$ of the restriction of H to $m_0 \times I$.

Claim. The trace of H at m_0 is in the subgroup $\pi_1(F, m_0)$.

Proof of Claim. When ∂M is nonempty, [21, Corollary 18.2] applies to prove the claim. When ∂M is empty, the argument in [21, Lemma 18.1] shows that if the claim is false then the components of V and $\overline{M - V}$

adjacent to F are each homeomorphic to the product of the torus and an interval. By maximality of V, this is only possible when M is a torus bundle over the circle which admits a Sol structure, which is excluded by hypothesis. This completes the proof of the claim.

Since F is a square, annulus, or torus, there is an isotopy on F from the identity to the identity whose trace is equal to the trace of H. So it is possible to change f by an admissible isotopy with support in a neighborhood of F, so that the trace of the isotopy from f to the identity of M is trivial. Also, f induces the identity automorphism on $\pi_1(F, m_0)$ and preserves each element of the boundary pattern of F (either the two boundary circles, if F is an annulus, or the four sides, if F is a square), so we may assume f is the identity map on F. At each element of the boundary pattern of F the trace must also be trivial. Using the analogue of Laudenbach's Theorem for simple closed curves and proper arcs in 2-manifolds, we may deform the isotopy admissibly so that each component of the boundary of F is preserved at each level of the isotopy. The trace at m_0 is still trivial, so using [26], we may assume that m_0 is actually fixed during the isotopy. Now Laudenbach's Theorem applies. This completes the proof of Proposition 4.1.1.

To avoid later distraction, we will deal now with the exceptional case described in Proposition 4.1.1.

Proposition 4.1.2. *Let M be a torus bundle over S^1 which admits a Sol structure. Then $\mathcal{H}(M)$ is finite.*

Proof. The characteristic submanifold Σ of M is a regular neighborhood of a fiber F. By [21, Corollary 27.6] the subgroup generated by Dehn twists about essential tori has finite index in $\mathcal{H}(M)$. Since every essential torus is isotopic into Σ, this subgroup is abelian and generated by Dehn twists about the fiber. But if ϕ is the attaching homeomorphism of the bundle, and x is an element in $\pi_1(F)$, then the Dehn twist about the fiber with trace x is isotopic to the Dehn twist with trace $\phi(x)$, hence the Dehn twist with trace $x^{-1}\phi(x)$ is nullisotopic. Since the matrix of $\phi_\#: \pi_1(F) \to \pi_1(F)$ has trace of absolute value at least three, the elements of the form $x^{-1}\phi(x)$ constitute a subgroup of finite index in $\pi_1(F)$. It follows that $\mathcal{H}(M)$ is finite.

4.2. The subgroup K of the subgroup $\mathcal{H}(M, \Sigma_1, \cdots, \Sigma_n, \underline{\underline{m}})$, and a surprising example. Let $(\Sigma_1, \underline{\underline{\sigma_1}}), (\Sigma_2, \underline{\underline{\sigma_2}}), \cdots, (\Sigma_n, \underline{\underline{\sigma_n}})$ be the components of $(\Sigma, \underline{\underline{\sigma}})$. Recall from §3.1 that the boundary pattern $\underline{\underline{\sigma_i}}$ consists of the components of the frontier of Σ_i in M, together with the components of the intersections of $\partial \Sigma_i$ with the elements of $\underline{\underline{m}}$. Clearly

$\mathscr{H}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ has finite index in $\mathscr{H}(M, \Sigma, \underline{m})$. Although it will not matter for us, the possibilities that can arise as the components of the closure $\overline{M - \Sigma}$ are described in [21, p. 159].

Lemma 4.2.1. *Assume that M is not a torus bundle over S^1 which admits a Sol structure. Then the image of the homomorphism $\mathscr{H}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m}) \to \mathscr{H}(\overline{M - \Sigma}, \mathrm{Fr}(\Sigma))$ induced by restriction is finite.*

Proof. By the argument in [21, Corollary 27.6], there is a subgroup of finite index in $\mathscr{H}(M, \Sigma, \underline{m})$ generated by Dehn twists about admissible essential tori and annuli in Σ. Thus, there is a subgroup of finite index in $\mathscr{H}(M, \Sigma, \underline{m})$ which can be represented by homeomorphisms which are the identity on $\overline{M - \Sigma}$. The lemma follows.

Notice in particular that when Σ is empty, the restriction in Lemma 4.2.1 is the identity, so Theorem 4.3.1 is proved for this case. When M is fibered (i.e., when $M = \Sigma$), Theorem 4.3.1 has been proved in §3. So for the remainder of §4, we will *assume that the characteristic submanifold Σ is not empty and is not equal to M*. In particular, the frontier of each component Σ_i of Σ is nonempty.

Define $\mathscr{K}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ to be the kernel of the homomorphism in Lemma 4.2.1. Recall the groups $\mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$ defined just before Lemma 3.6.2.

Lemma 4.2.2. *There is a surjective homomorphism*

$$\rho: \mathscr{K}\left(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m}\right) \to \prod \mathscr{G}\left(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}\left(\Sigma_i\right)\right),$$

whose kernel is the finitely generated abelian subgroup K generated by Dehn twists about the components of the frontier of Σ.

Proof. Since the frontier of each Σ_i is nonempty, the argument of [19, Lemma VI.19] shows that each element of $\mathscr{K}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ is representable by a homeomorphism whose restriction to each $(\Sigma_i, \underline{\sigma_i})$ is fiber-preserving. According to Theorem 3.5.1, this fiber-preserving homeomorphism is unique up to admissible fiber-preserving isotopy. Therefore the restriction homomorphism ρ is well defined. Since the elements of $\mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$ have representatives which are the identity on the frontier of Σ_i, ρ is surjective. Any element of the kernel of ρ is isotopic to a homeomorphism which is the identity outside a neighborhood of the components of the frontier of Σ, giving the description of K.

Lemma 4.2.2 gives us the following reduction of Theorem 4.3.1.

Proposition 4.2.3. *If $\mathscr{K}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ contains a subgroup of finite index, which intersects K in a torsion-free subgroup, then $\mathscr{H}(M, \underline{m})$ is a finitely presented virtual duality group of type VFL.*

Proof. If M has a *Sol* structure, then $\mathscr{H}(M)$ is finite, and has the desired properties. So assume that Proposition 4.1.1 applies to M. By Lemma 3.6.2, $\prod \mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$ is a finitely presented virtual duality group of type VFL. Hence $\overline{\overline{\mathscr{H}(M}}, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ contains a subgroup of finite index, whose image under ρ is a finitely presented virtual duality group of type FL. The intersection of this subgroup with a subgroup of finite index which intersects K in a torsion-free subgroup is an extension of a finitely generated free abelian group by a finitely presented duality group of type FL, hence is the desired subgroup of $\mathscr{H}(M, \underline{m})$.

Example 4.2.4. It may seem surprising that K can contain torsion. To construct an example, let Σ_1 and Σ_2 be Seifert fibered, each with one boundary component and an orientable orbit space having negative Euler characteristic. Let $t_i^{1,0}$ and $t_i^{0,1}$ be the isotopy classes of Dehn twists about $\partial \Sigma_i$ with traces respectively the fiber z_i and a cross section w_i to the fibering on $\partial \Sigma_i$. Form $M(a, b, c, d)$ from $\Sigma_1 \cup \Sigma_2$ by identifying $\partial \Sigma_1$ to $\partial \Sigma_2$ using a homeomorphism that sends z_1 to $az_2 + bw_2$ and w_1 to $cz_2 + dw_2$, where $ad - bc = 1$ and $b \neq 0$. Since the orbit spaces are orientable, Lemma 3.5.3 shows that the $t_i^{1,0}$ are isotopic to the identity (rel $\partial \Sigma_i$). Since the trace z_1 of $t_1^{0,1}$ is identified to $az_2 + bw_2$, it follows that $t_1^{1,0}$ is isotopic as a homeomorphism of M to $-at_2^{1,0} - bt_2^{0,1}$, and a similar relation occurs for $t_1^{0,1}$. By Laudenbach's Theorem in §4.1, if a homeomorphism of M preserving $\partial \Sigma_1$ is isotopic to the identity, then it is isotopic to the identity preserving $\partial \Sigma_1$. Putting these observations together, we have the following abelian presentation for K:

$$\left\langle t_1^{1,0}, t_1^{0,1}, t_2^{1,0}, t_2^{0,1} \mid t_1^{1,0} = 0,\, t_2^{1,0} = 0,\, t_1^{1,0} + at_2^{1,0} + bt_2^{0,1} = 0, \right.$$
$$\left. t_1^{0,1} + ct_2^{1,0} + dt_2^{0,1} = 0 \right\rangle$$

from which one finds that $K \cong \mathbb{Z}/b$ generated by $t_2^{0,1}$.

Remark 4.2.5. I am fairly certain that in general, by carefully analyzing a presentation matrix for K, it can be shown that K is torsion-free unless M is a graph manifold [38] in which the fundamental group of each component of Σ has infinite cyclic center. The simplification of the remainder of our proof of Theorem 4.3.1 that would result from the use of this information is minor and would not justify the work entailed in the analysis of K.

Remark 4.2.6. In Lemma 9.5 we will compute the rank of K, as a step in the calculation of the virtual cohomological dimension of the mapping class group.

4.3. The proof of the main theorem. By Proposition 4.2.3, Theorem 4.3.1, stated at the beginning of §4, is immediate from the following theorem.

Theorem 4.3.2. $\mathscr{H}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ *contains a subgroup of finite index, which intersects* K *in a torsion-free subgroup.*

To prove Theorem 4.3.2, we will construct a homomorphism from a subgroup of finite index in $\mathscr{H}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ to a finite group, which injects on the torsion of K. The kernel of this homomorphism is the subgroup sought in Theorem 4.3.2.

The construction of the homomorphism will use the following property of surfaces.

Lemma 4.3.3. *Let* $(F, \underline{f})$ *be a connected surface of finite type with* $\chi(F) < 0$, *and let* $\underline{f_1}$ *be a nonempty subset of* $\underline{f}$. *Let* $W_1, W_2, \cdots, W_n$ *be the elements of* $\underline{f_1}$ *that are boundary components of* F, *and let* t_{W_i} *denote the isotopy class of a Dehn twist about* W_i. *Let* b *be an integer with* $b \geq 3$. *Then for some* m, *there are a subgroup* $\mathscr{L}(F, \underline{f} \text{ rel } |\underline{f_1}|)$ *of finite index in* $\mathscr{H}_+(F, \underline{f} \text{ rel } |\underline{f_1}|)$ *and a homomorphism*

$$\phi \colon \mathscr{L}\left(F, \underline{f} \text{ rel } \left|\underline{f_1}\right|\right) \to \mathrm{GL}(m, \mathbb{Z})$$

with the following properties:

(1) $\mathscr{L}(F, \underline{f} \text{ rel } |\underline{f_1}|)$ *contains the* t_{W_i}.

(2) *The elements of* $\mathscr{L}(F, \underline{f} \text{ rel } |\underline{f_1}|)$ *act trivially on* $H_1(F, |\underline{f_2}|; \mathbb{Z}/b)$.

(3) *The elements* $\{\phi(t_{W_i})\}$ *form a basis for a subgroup which is isomorphic to* $\mathbb{Z}^n$.

(4) *Upon passage to* $\mathrm{GL}(m, \mathbb{Z}/b)$, *the elements* $\{\phi(t_{W_i})\}$ *form a basis for a subgroup which is isomorphic to* $(\mathbb{Z}/b)^n$.

Proof. Let $\underline{f_2} = \underline{f} - \underline{f_1}$. We first consider the case when F is orientable. Suppose that $n \geq 3$. Suppose first that $\bigcup W_i = \partial F$. Let F_1 be a homeomorphic copy of F and let $D(F) = F \cup_{\partial F} F_1$ be the double of F. Extending by the identity defines a homomorphism $\mathscr{H}(F \text{ rel } \partial F) \to \mathscr{H}(D(F))$. Let $N = H_1(D(F); \mathbb{Z})$. Choose for N a basis of the form

$$\left\{ A_1, A_2, \cdots, A_{n-1}, W_1, W_2, \cdots, W_{n-1}, X_1, X_2, \cdots, X_r \right\},$$

where each A_i intersects W_i and W_n each in one point and is disjoint from all other W_j, and where each X_i can be represented by a cycle disjoint from ∂F. Let E_i denote the $(n-1) \times (n-1)$ matrix with 1 in

the (i, i)-entry and zeros in all the other entries. Let I_k denote the $k \times k$ identity matrix, and $0_{k,l}$ the $k \times l$ zero matrix. For $1 \leq i \leq n-1$, we have on N that

$$\left(t_{W_i}\right)_* = \begin{bmatrix} I_{n-1} & 0_{n-1,n-1} & 0_{n-1,r} \\ E_i & I_{n-1} & 0_{n-1,r} \\ 0_{r,n-1} & 0_{r,n-1} & I_r \end{bmatrix}$$

while

$$\left(t_{W_n}\right)_* = \begin{bmatrix} I_{n-1} & 0_{n-1,n-1} & 0_{n-1,r} \\ A & I_{n-1} & 0_{n-1,r} \\ 0_{r,n-1} & 0_{r,n-1} & I_r \end{bmatrix}$$

where A is the $(n-1) \times (n-1)$ matrix with every entry equal to -1. In this case, define $\mathscr{L}(F, \underline{f} \text{ rel } |\underline{f_1}|)$ to be the kernel of $\mathscr{H}_+(F, \underline{f} \text{ rel } |\underline{f_1}|) \to$ $\text{Aut}(H_1(F, |\underline{f_2}|; \mathbb{Z}/b))$. Putting $m = \dim(N)$, Lemma 4.3.3 follows for the case $\bigcup \overline{W_i} = \partial F$ when ϕ is taken to be the composite

$$\mathscr{L}\left(F, \underline{f} \text{ rel } \left|\underline{f_1}\right|\right) \to \mathscr{H}\left(D(F)\right) \to \text{Aut}\left(H_1(D(F))\right) \cong \text{GL}(m, \mathbb{Z}).$$

Now suppose that $\bigcup W_i \neq \partial F$ (and still $n \geq 3$). This time form the double along $\bigcup W_i$. Extending by the identity yields a homomorphism

$$\mathscr{H}\left(F, \underline{f} \text{ rel } \left|\underline{f_1}\right|\right) \to \mathscr{H}\left(D(F), \underline{f} - \{W_i\} \text{ rel } \left(\left|\underline{f_1}\right| - \bigcup W_i\right)\right).$$

This time, choosing for $H_1(D(F))$ a basis of the form

$$\left\{A_1, A_2, \cdots, A_{n-1}, W_1, W_2, \cdots, W_n, X_1, X_2, \cdots, X_r\right\},$$

the lemma follows in a manner similar to the previous case.

If $n = 2$ and $\chi(F) = -1$ (so F is a disc-with-two-holes) then the argument just given still works.

Suppose $n \leq 2$ and, if $n = 2$, then F is not a disc-with-two-holes. Let $(\mathbb{Z}/3)^q$ be the quotient of $H_1(F; \mathbb{Z}/3)$ by the subgroup generated by the homology classes of the W_i. Since $\chi(F) < 0$ and F is not a disc-with-two-holes, it follow that $q \geq 1$. Let $\widetilde{F}$ be the covering of F corresponding to the kernel of the composite $\pi_1(F) \to H_1(F; \mathbb{Z}/3) \to (\mathbb{Z}/3)^q$. Observe that

1. The kernel is preserved by the induced automorphism of any element of $\mathscr{H}(F, \underline{f} \text{ rel } |\underline{f_1}|)$, so each such element has lifts to $\widetilde{F}$.

2. Each component of the preimage of any element of $\underline{f}$ projects homeomorphically to W_i; in particular, the preimage of $\bigcup \overline{W_i}$ consists of $3^q n \geq 3$ circles.

3. The preimages of the elements of the boundary patterns on F determine boundary patterns $\underline{\tilde{f}}$, $\underline{\tilde{f}_1}$, and $\underline{\tilde{f}_2}$ on $\tilde{F}$.

Define $\mathscr{L}(F, \underline{f} \text{ rel } |\underline{f_1}|)$ to be the set of elements $\langle h \rangle$ in the kernel of $\mathscr{H}(F, \underline{f} \text{ rel } |\underline{f_1}|) \rightarrow \text{Aut}(\overline{H_1}(F, |\underline{f_2}|; \mathbb{Z}/b))$ for which some (hence exactly one) lift $\tilde{h}$ of h preserves each element of $\underline{\tilde{f}}$. Observe that

4. Lifting $\langle h \rangle$ to $\langle \tilde{h} \rangle$ defines a homomorphism from $\mathscr{L}(F, \underline{f} \text{ rel } |\underline{f_1}|)$ to $\mathscr{H}(\tilde{F}, \underline{\tilde{f}} \text{ rel } |\underline{\tilde{f}_1}|)$.

5. Each t_{W_i} lifts to a product of Dehn twists about the preimages of W_i.

Composing the lifting homomorphism with the homomorphism

$$\mathscr{H}\left(\tilde{F}, \underline{\tilde{f}} \text{ rel } \left|\underline{\tilde{f}_1}\right|\right) \rightarrow \text{GL}(m, \mathbb{Z}/b)$$

obtained for the case $n \geq 3$ yields the homomorphism needed.

In case F is nonorientable, let $\hat{F}$ be the orientable double cover of F. Choosing the orientation-preserving lift defines a homomorphism from $\mathscr{H}(F, \underline{f} \text{ rel } |\underline{f_1}|)$ to $\mathscr{H}(\hat{F}, \underline{\hat{f}} \text{ rel } |\underline{\hat{f}_1}|)$. Passing to appropriate subgroups and using the homomorphism to $\overline{\text{GL}}(m, \mathbb{Z})$ already constructed for the orientable case completes the proof of Lemma 4.3.3.

The next task is to produce a 3-dimensional version of ϕ. To simplify notation during the next lemma, write Σ for a single component of the characteristic submanifold of M. Let $\underline{\sigma_1}$ be the boundary pattern consisting of the components of $\text{Fr}(\Sigma)$, and let $\underline{\sigma_2} = \underline{\sigma} - \underline{\sigma_1}$. Since M is not fibered, $\underline{\sigma_1}$ is nonempty. Denote by F the orbit surface of Σ, and by $\underline{f}$, $\underline{f_1}$, and $\underline{f_2}$ the images of $\underline{\sigma}$, $\underline{\sigma_1}$, and $\underline{\sigma_2}$ in F.

Suppose first that Σ is Seifert fibered. Write $T_1, T_2, \cdots, T_k$ for the elements of $\underline{\sigma_1}$ that are tori, and $A_1, A_2, \cdots, A_l$ for the elements that are annuli. Let $t_j^{1,0}$ denote the isotopy classes in $\mathscr{H}_+(\Sigma, \underline{\sigma} \text{ rel } |\underline{\sigma_1}|)$ of Dehn twists about T_j with traces the fiber of the Seifert fibering. If F is orientable, choose the $t_j^{1,0}$ to have equal traces with respect to a global orientation of the fiber. If F is nonorientable, choose the $t_j^{1,0}$ to have equal traces with respect to a global orientation of the fiber of the Seifert manifold obtained by removing from M the preimage of the center circles of the crosscaps of F. Let $t_j^{0,1}$ denote the isotopy classes of Dehn twists about T_j whose traces are the intersection with T_j of a cross section to the fibering defined over the complement of the exceptional points in F. If F is an annulus with no exceptional points and $\underline{\sigma_1}$ consists of two tori,

choose $t_1^{0,1}$ equal to the inverse of $t_2^{0,1}$. Let a_i denote the isotopy classes of Dehn twists about the A_i, again with compatible traces.

If Σ is an I-bundle, let $t_j^{0,1}$, $1 \le j \le k$, denote the isotopy classes of Dehn twists about the annuli in $\underline{\sigma}_1$. If F is an annulus with no exceptional points, and $\underline{\underline{\sigma}}_1$ consists of two annuli, choose $t_1^{0,1}$ equal to the inverse of $t_2^{0,1}$.

We are now prepared for the main technical lemma to be used in the proof of Theorem 4.3.2.

Lemma 4.3.4. (a) *The elements* $\{t_1^{1,0}, \cdots, t_k^{1,0}, t_1^{0,1}, \cdots, t_k^{0,1}, a_1,$ $\cdots, a_l\}$ *generate a free abelian subgroup* A *of* $\mathcal{H}_+(\Sigma, \underline{\sigma} \text{ rel } |\underline{\underline{\sigma}}_1|)$. *If* F *is orientable, then these elements satisfy the relation* $\Sigma_{i=1}^k t_i^{1,0} + \Sigma_{j=1}^l a_j = 0$. *If* F *is a Möbius band with no exceptional points and* $\underline{f}_1 = \{\partial F\}$, *or if* F *is an annulus with no exceptional points and either* $\underline{\underline{f}}_1$ *consists of both boundary components of* F, *or* $\underline{f}_1$ *consists of one boundary component of* F *and no element of* $\underline{f}$ *is an arc, then the relation* $\Sigma_{i=1}^k t_i^{0,1}$ *is satisfied. All other relations among these generators are consequences of these.*

(b) *Let* b *be an integer with* $b \ge 3$. *Then for some* r *and* m, *there is a subgroup* $\mathcal{L}(\Sigma, \underline{\sigma} \text{ rel } |\underline{\underline{\sigma}}_1|)$ *of finite index in* $\mathcal{H}_+(\Sigma, \underline{\sigma} \text{ rel } |\underline{\underline{\sigma}}_1|)$ *which contains* A *and admits a homomorphism*

$$\Phi \colon \mathcal{L}\left(\Sigma, \underline{\sigma} \text{ rel } \left|\underline{\underline{\sigma}}_1\right|\right) \to (\mathbb{Z}/b)^r \times \mathrm{GL}(m, \mathbb{Z}/b)$$

whose kernel intersects A *in the subgroup* bA.

Proof. Suppose first that Σ is an I-bundle, so that

$$\mathcal{H}_+\left(\Sigma, \underline{\sigma} \text{ rel } \left|\underline{\underline{\sigma}}_1\right|\right) \cong \mathcal{H}_+\left(F, \underline{f} \text{ rel } \left|\underline{\underline{f}}_1\right|\right)$$

by Proposition 3.2.1. The elements $\{t_1^{0,1}, \cdots, t_k^{0,1}\}$ correspond to Dehn twists about the boundary components of F that are elements of $\underline{f}_1$. When F is a disc, no element of $\underline{\sigma}$ is an annulus (because the boundary pattern of Σ is useful, there must be at least four arcs in $\underline{f}$ if F is a disc), so there is nothing to prove. If F is a Möbius band with no exceptional points, then $\mathcal{H}_+(\Sigma, \underline{\sigma} \text{ rel } |\underline{\underline{\sigma}}_1|)$ is trivial and the lemma is obvious. Suppose F is an annulus with no exceptional points. If one of the boundary components of F contains no element of $\underline{f}_1$, and no element of $\underline{f}$ that is an arc, then $\mathcal{H}_+(\Sigma, \underline{\sigma} \text{ rel } |\underline{\underline{\sigma}}_1|)$ is trivial and the lemma is obvious. Otherwise $\mathcal{H}_+(\Sigma, \underline{\sigma} \text{ rel } |\underline{\underline{\sigma}}_1|) \cong \mathbb{Z}$ and projection to $\mathbb{Z}/b$ will satisfy the conclusion of Lemma 4.3.4.

In the remaining cases, the elements $\{t_1^{0,1}, \cdots, t_k^{0,1}\}$ form a basis for a free abelian subgroup of $\mathscr{H}_+(F, \underline{f} \text{ rel } |f_1|)$, so (a) is proved for I-bundles. Let $\mathscr{L}(\Sigma, \underline{\sigma} \text{ rel } |\sigma_1|)$ be the subgroup of $\mathscr{H}_+(\Sigma, \underline{\sigma} \text{ rel } |\sigma_1|)$ that corresponds to the subgroup $\mathscr{L}(F, \underline{f} \text{ rel } |f_1|)$ in Lemma 4.3.3 (under the isomorphism of Proposition 3.2.1). Using Lemma 4.3.3 produces a composite

$$\mathscr{L}\left(\Sigma, \underline{\sigma} \text{ rel } \left|\sigma_1\right|\right) \to \mathscr{L}\left(F, \underline{f} \text{ rel } \left|f_1\right|\right) \to \text{GL}(m, \mathbb{Z}/b)$$

satisfying the conclusion of Lemma 4.3.4(b).

From now on, we assume that Σ is Seifert fibered. As for I-bundles, the cases when F is a Möbius band or annulus with no exceptional points can be handled by inspection, so we assume that $\chi(F - \mathscr{E}) < 0$. Write B for the free abelian group $\mathscr{H}_T^0(\Sigma, \underline{\sigma} \text{ rel } |\sigma_1|)$ studied in §3.5. From Proposition 3.5.8(b) we have $B \circ \mathscr{H}^*(F, \underline{f} \text{ rel } |f_1|)$ as a subgroup of finite index in $\mathscr{H}_+(\Sigma, \underline{\sigma} \text{ rel } |\sigma_1|)$, such that the elements $\{t_1^{1,0}, \cdots, t_k^{1,0}, a_1, \cdots, a_l\}$ generate a summand B' of B, subject to the relation specified in Lemma 3.5.6. The Dehn twists $\{t_1^{0,1}, \cdots, t_k^{0,1}\}$ generate a free abelian subgroup C of $\mathscr{H}^*(F, \underline{f} \text{ rel } |f_1|)$ that commutes with all vertical Dehn twists, so Lemma 4.3.4(a) is proved for the Seifert fibered case.

Apply Lemma 4.3.3 to $(F - \mathscr{E}, \underline{f} \text{ rel } |f_1|)$, obtaining for some m a homomorphism $\varphi \colon \mathscr{L}^*(F - \mathscr{E}, \underline{f} \text{ rel } |f_1|) \to \text{GL}(m, \mathbb{Z}/b)$ carrying C to C/bC. Regarding $\mathscr{H}^*(F, \underline{f} \text{ rel } |f_1|)$ as a subgroup of finite index in $\mathscr{H}(F - \mathscr{E}, \underline{f} \text{ rel } |f_1|)$, define $\mathscr{L}^*(F, \underline{f} \text{ rel } f_1|)$ to be the intersection of $\mathscr{H}^*(F, \underline{f} \text{ rel } |f_1|)$ with $\mathscr{L}(F - \mathscr{E}, \underline{f} \text{ rel } |f_1|)$. Let $\mathscr{L}(\Sigma, \underline{\sigma} \text{ rel } |\sigma_1|)$ denote $B \circ \mathscr{L}^*(F, \underline{f} \text{ rel } |f_1|)$. Note that C is contained in $\mathscr{L}(\Sigma, \underline{\sigma} \text{ rel } |\sigma_1|)$. The subgroup bB is characteristic in B, hence normal in $\mathscr{L}(\Sigma, \underline{\sigma} \text{ rel } |\sigma_1|)$. Since $\mathscr{L}^*(F, \underline{f} \text{ rel } |f_1|)$ acts trivially on B modulo b, the quotient of $\mathscr{L}(\Sigma, \underline{\sigma} \text{ rel } |\sigma_1|)$ by bB is a direct product $(B/bB) \times \mathscr{L}^*(F, \underline{f} \text{ rel } |f_1|)$. Letting Φ be the composite

$$\mathscr{L}\left(\Sigma, \underline{\sigma} \text{ rel } \left|\sigma_1\right|\right) \to (B/bB) \times \mathscr{L}^*\left(F, \underline{f} \text{ rel } \left|f_1\right|\right)$$

$$\xrightarrow{\quad 1_{B/bB} \times \varphi \quad} B/bB \times \text{GL}(m, \mathbb{Z}/b)$$

completes the proof of Lemma 4.3.4.

We can now complete the proof of Theorem 4.3.2, and hence of Theorem 4.3.1. Let $b \geq 3$ be an integer divisible by the order of all torsion in the finitely generated abelian group K. For each i, write $\mathscr{L}_i$ for $\mathscr{L}(\Sigma_i, \underline{\underline{\sigma}}_i \text{ rel } \text{Fr}(\Sigma_i))$ and let A_i and $\Phi_i \colon \mathscr{L}_i \to (\mathbb{Z}/b)^{r_i} \times \text{GL}(m_i, \mathbb{Z}/b)$ be the corresponding A and Φ from Lemma 4.3.4. Note that $\prod A_i$ is central in $\prod \mathscr{L}_i$ (because all elements of $\mathscr{L}_i$ are the identity on $\text{Fr}(\Sigma_i)$).

Extending by the identity defines a surjective homomorphism

$$\prod_{i=1}^{n} \mathscr{H}_+ \left(\Sigma_i, \underline{\underline{\sigma}}_i \text{ rel } \text{Fr}(\Sigma_i) \right) \to \mathscr{H} \left(M, \Sigma_1, \cdots, \Sigma_n, \underline{\underline{m}} \right).$$

Let e denote its restriction to $\prod_{i=1}^{n} \mathscr{L}_i$. Since $\mathscr{L}_i$ has finite index in $\mathscr{H}(\Sigma_i \underline{\underline{\sigma}}_i \text{ rel } \text{Fr}(\Sigma_i))$, the image of e has finite index. Now $\ker(e) \subseteq \prod A_i$, since any homeomorphism of a Σ_i which is the identity on $\partial \Sigma_i$ and is admissibly isotopic to the identity on Σ_i is isotopic (rel $\partial \Sigma_i$) to a product of Dehn twists about the components of $\text{Fr}(\Sigma_i)$. Since the Dehn twists about the components of $\text{Fr}(\Sigma)$ generate K, it follows that $K = e(\prod A_i) \cong (\prod A_i)/\ker(e)$.

Now put

$$\Psi = \prod_{i=1}^{n} \Psi_i \colon \prod_{i=1}^{n} \mathscr{L}_i \to \prod_{i=1}^{n} (\mathbb{Z}/b)^{r_i} \times \text{GL}\left(m_i, \mathbb{Z}/b\right).$$

Because of the properties of the Φ_i given in Lemma 4.3.4, $\ker(\Psi) \cap \prod A_i = \prod b A_i$. Now Ψ induces a homomorphism from image(e) to image(Ψ)/$\Psi(\ker(e))$, which carries K onto

$$\Psi(\prod A_i)/\Psi(\ker(e)) \cong \prod A_i/(\ker(e) + \prod b A_i).$$

If b is chosen so that the least common multiple of the torsion in K divides b, then this induced homomorphism will be injective on the torsion of K. For if an element of $\prod A_i$ is divisible by b, it is either already trivial in K, or it has infinite order in K. This completes the proof of Theorem 4.3.2 and hence of Theorem 4.3.1.

Corollary 4.3.5. *Let M be a Haken 3-manifold. Then $\mathscr{H}(M)$ is a finitely presented virtual duality group of type VFL.*

Proof. Since M is Haken, the boundary pattern $\underline{\partial M}$ is complete and useful. Since $\mathscr{H}(M, \underline{\partial M})$ has finite index in $\mathscr{H}(M)$, the result follows from Theorem 4.3.1.

By a trick, we can easily obtain a relative version of Theorem 4.3.1.

Theorem 4.3.6. *Let $(M, \underline{\underline{m}})$ be a compact orientable irreducible sufficiently large 3-manifold with complete and useful boundary pattern, and*

let $S \subseteq \partial M$ be a submanifold which is the union of some elements of $\underline{m}$. Then $\mathscr{H}(M, \underline{m} \text{ rel } S)$ is a finitely presented virtual duality group of type VFL.

Proof. Form from $\underline{m}$ a new boundary pattern for M by replacing the elements which make up S by the closed 2-cells of the cell structure dual to a fine triangulation of S. There is a deformation retraction from the space of homeomorphisms that preserve each element of the new boundary pattern to the space of homeomorphisms that are the identity on S. Apply Theorem 4.3.1 to the corresponding mapping class group.

Corollary 4.3.7. *Let M be a compact orientable irreducible sufficiently large 3-manifold. Then $\mathscr{H}(M \text{ rel } \partial M)$ is a finitely presented virtual duality group of type VFL.*

Proof. The 2-cells of a cell structure dual to a fine triangulation of ∂M form a complete and useful boundary pattern. Now apply Theorem 4.3.6 with $S = \partial M$.

5. The disc complex

Let M be an irreducible 3-manifold with compressible boundary. By a *disc in M* we mean a properly imbedded 2-disc $(D, \partial D) \subseteq (M, \partial M)$. The disc is *essential* when ∂D does not bound a 2-disc in ∂M. Equivalently, D is not parallel to a disc in ∂M. Define the *disc complex* of M to be the simplicial complex L whose vertices are the isotopy classes of properly-imbedded essential 2-discs in M, and whose simplices are determined by the rule that a collection of $n + 1$ distinct vertices spans an n-simplex if and only if it admits a collection of representatives which are pairwise disjoint. This is a modification of the complex used in [23], and some of the ideas of this section are derived from that paper.

Let $[D]$ and $[E]$ be vertices of L. Define $[D] \cdot [E]$ to be one-half of the minimal cardinality of $\partial D' \cap \partial E'$, where D' and E' are isotopic to D and E, respectively, and intersect transversely.

Lemma 5.1. *Let $D_0, D_1, \cdots, D_n$ be essential discs in M. Then there are discs D_i' isotopic to D_i such that for all $i \neq j$, $D_i' \cap D_j'$ consists of $[D_i] \cdot [D_j]$ arcs.*

Proof. We may assume that the boundaries of the discs are pairwise nonisotopic and all lie in a single boundary component F of M. If $\chi(F) = 0$, then at most one essential simple closed curve in F bounds an imbedded disc in M, so the result follows. So we assume that $\chi(F) < 0$ and hence F can be given a complete hyperbolic structure. The boundaries of the D_i are essential 2-sided simple closed curves, hence are

isotopic to unique imbedded geodesic loops, which intersect pairwise in the minimum number of points for any loops in their isotopy classes in F. Choose a convex Riemannian metric for M. Using Theorem 6 of [29], these discs are isotopic, fixing their boundaries, to a collection of discs which are minimal surfaces. These discs cannot intersect in simple closed curves, so Lemma 5.1 is proved.

As a special case, we have

Lemma 5.2. *Let* $\{v_0, v_1, \cdots, v_n\}$ *be a collection of distinct vertices in* L. *If for each* $i \neq j$, v_i *and* v_j *bound a 1-simplex in* L, *then* $\{v_0, v_1, \cdots, v_n\}$ *spans an n-simplex in* L.

Proof. Since $v_i \cdot v_j = 0$ for all i and j, Lemma 5.1 implies that representatives for the vertices can be chosen to be disjoint.

Theorem 5.3. *Let* M *be an irreducible 3-manifold with compressible boundary. Then the disc complex* L *of* M *is contractible.*

Proof. Since L is a CW complex, it suffices to prove that the homotopy groups are trivial. Choose as basepoint a vertex k_0. Let $f: S^q \to L$, $q \geq 0$, be any map carrying the basepoint of S^q to k_0. We may assume that f is simplicial with respect to some triangulation K of S^q. Define the complexity of the pair (f, K) as follows. Let $\mathbb{Z}_{\geq 0}$ denote the nonnegative integers. For $i \geq 0$, define $C_i(f, K)$ to be the number of vertices $v \in K$ such that $f(v) \cdot k_0 = i$. The complexity $C(f, K)$ is the element

$$\left(\cdots, C_2(f, K), C_1(f, K), C_0(f, K)\right) \in \sum_{i=\infty}^{0} \mathbb{Z}_{\geq 0}.$$

The complexities are ordered lexicographically.

If $C_i(f, K) = 0$ for all $i \geq 1$, then Lemma 5.2 shows that the image of f lies in the closed star $\mathrm{st}(k_0)$ and hence f is nullhomotopic. So we assume that $C_i(f, K) = 0$ for all $i > n$ and that $C_n(f, K) > 0$ for some $n > 0$. Choose a vertex $v \in K$ such that $f(v) \cdot k_0 = n$. Let $v_1, v_2, \cdots, v_s$ be the vertices in K adjacent to v. Choose representatives D_i for $f(v_i)$, D for $f(v)$, and E_0 for the basepoint k_0 so that the collection $\{D_1, D_2, \cdots, D_s, D, E_0\}$ satisfies the conclusion of Lemma 5.1. Then each D_i is disjoint from D, and discs representing vertices that are adjacent in $\mathrm{st}(v)$ are disjoint.

Since $n > 0$, D is not disjoint from E_0. Consider an arc of their intersection α which is outermost on E_0. There is a disc $B \subseteq E_0$ such that $\partial B \subseteq \alpha \cup \partial E_0$ in B contains no other components of $D \cap E_0$. The boundary of a regular neighborhood N of $D \cup B$ consists of three properly-imbedded discs, one of which is parallel in N to D and the other

two having fewer arcs of intersection with E_0 than D had. At least one of these two discs must be essential (in fact, both are when ∂D and ∂E_0 intersect minimally) and an essential disc obtained in this way is said to result from *surgery on D along B*.

Suppose for now that $B \cap (\bigcup_{i=1}^{s} D_i)$ is empty. Let D' result from surgery on D along B. Now D' is disjoint from D and from all D_i (it may be isotopic to some D_i) so Lemma 5.2 implies that L contains the join $[D'] * \mathrm{st}([D])$. Let $g: K \to L$ be the simplicial map defined on vertices by $g(w) = f(w)$ if $w \neq v$ and $g(v) = [D']$. Since L contains $[D'] * \mathrm{st}([D])$, g is homotopic to f. Since $[D'] \cdot [E_0] < [D] \cdot [E_0]$, we have $C(g, K) < C(f, K)$.

Suppose now that $B \cap (\bigcup_{i=1}^{s} D_i)$ is nonempty. Let β be an arc of intersection of $B \cap D_k$ which is outermost in B and contains no arc of intersection of any D_i which is disjoint from D_k (note that β may still intersect an arc of some $D_i \cap B$). Let D'_k result from surgery on D_k along the disc in E_0 cut off by β. Construct a subdivision K' of K by adding the barycenter v_0 of the simplex $\langle [D], [D_k] \rangle$ as a vertex (i.e., replace each simplex of the form $\langle [D], [D_k], w_1, w_2, \cdots, w_r \rangle$ by the two simplices $\langle [D], v_0, w_1, w_2, \cdots, w_r \rangle$ and $\langle v_0, [D_k], w_1, w_2, \cdots, w_r \rangle)$. Define the simplicial map $f': K' \to L$ on vertices by $f'(v_0) = [D'_k]$ and $f'(w) = f(w)$ for all other vertices. Since D'_k is disjoint from D and from all the D_i that were disjoint from D_k, the join $[D'_k] * \mathrm{st}(\langle [D], [D_k] \rangle)$ is contained in L, and therefore f' is homotopic to f. Now $C(f', K') > C(f, K)$, because we have added the new vertex v_0 mapping to $[D'_k]$, but $[D'_k] \cdot [E_0] < [D_k] \cdot [E_0] \leq n$, since n was maximal, so $C_m(f', K') = C_m(f, K)$ for all $m \geq n$, and the discs that represent the images of the vertices of $\mathrm{st}_{K'}(v)$ in L have fewer arcs of intersection with B than before. Repeating finitely many times, we obtain a subdivision K'' of K and a simplicial map $f'': K'' \to L$ homotopic to f so that $C_m(f'', K'') = C_m(f, K)$ for $m \geq n$, and $\mathrm{int}(B)$ is disjoint from the representative discs for the images of the vertices of $\mathrm{st}_{K''}(v)$. Now, surgery on D along B yields as above a simplicial map $g: K'' \to L$ homotopic to f such that $C_m(g, K'') = C_m(f, K) = 0$ for $m > n$ and $C_n(g, K'') < C_n(f, K)$, hence $C(g, K'') < C(f, K)$. Induction completes the proof.

For use in §8, a couple of variations are needed. Define the *nonseparating disc complex L'* to be the full subcomplex of L spanned by the vertices whose representatives do not separate M (a collection of discs representing a simplex of L' may separate M; it is only required that each individual disc in the collection is nonseparating).

Theorem 5.4. L' *is contractible.*

Proof. The proof is the same as the proof of Theorem 5.3. The only observation necessary is that when surgery is performed on a nonseparating disc, at least one of the two discs that result from the surgery is nonseparating.

Theorem 5.5.   *Let* V_2 *be the orientable handlebody of genus* 2, *and let* $\langle[D]\rangle$ *be a vertex of the nonseparating disc complex* L' *of* V_2. *Then the link of* $\langle[D]\rangle$ *in* L' *is contractible.*

Proof The link of $\langle[D]\rangle$ is the full subcomplex of L' spanned by the vertices represented by discs that are disjoint from and not parallel to D. The proof that this 1-complex is contractible is the same as the proof of Theorem 5.3. The only observation necessary is that when surgery is performed on a nonseparating disc which is disjoint from and not parallel to D, at least one of the two discs that result from surgery is nonseparating and not parallel to D. This is a special property of the genus 2 handlebody.

6. Mapping class groups of products-with-handles

Let V be an orientable product-with-handles; that is, a connected manifold which can be constructed from the product of a (not necessarily connected) compact orientable aspherical 2-manifold B by forming $B \times I$ and then attaching a finite number of 1-handles to $B \times \{1\}$. These are extensively studied in [3], [28].

A handlebody is the special case of a product-with-handles where each component of B has nonempty boundary. We allow the set of 1-handles to be empty, so another special case is that of the product of a closed connected surface with I.

The component of ∂V which has nonempty intersection with $B \times \{1\}$ is denoted by F. It will be compressible except when V is the product of a closed surface with I. The rest of the boundary of V consists of incompressible components and is denoted by $\partial_0 V$.

Let $\mathrm{GHomeo}(V)$ be the group of orientation-preserving homeomorphisms of V which preserve F, whose restriction to $\partial_0 V$ is isotopic to the identity, and which induce the identity automorphism on $H_1(V; \mathbb{Z}/15)$. This consists of path components of $\mathrm{Homeo}(V)$ which form a normal subgroup of $\mathscr{H}(V)$ denote by $\mathscr{G}(V)$.

Theorem 6.1.   *Let* V *be a product-with-handles. Then* $\mathscr{G}(V)$ *is finitely presented and of type* FL.

The proof will occupy the remainder of this section.

Lemma 6.2. $\mathscr{G}(V)$ *is torsion-free.*

Proof. Suppose $\langle g \rangle \in \mathscr{G}(V)$ has finite order, and suppose for now that V is a handlebody. Choose a basis $\{a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g\}$ for $H_1(F; \mathbb{Z}/15)$ so that the intersection number $a_i \cdot b_j$ is equal to δ_{ij} and all $a_i \cdot a_j = b_i \cdot b_j = 0$, so that all b_i equal 0 in $H_1(V; \mathbb{Z}/15)$, and so that $\{a_1, a_2, \cdots, a_g\}$ is a basis for $H_1(V; \mathbb{Z}/15)$. Since $g|_F$ extends to V, the matrix A of $(g|_F)_*$ on $H_1(\partial V; \mathbb{Z}/15)$ has the form

$$ \mathrm{A} = \begin{bmatrix} \mathrm{I}_g & 0 \\ \mathrm{X} & \mathrm{Y} \end{bmatrix}. $$

For all i and j, we have $\delta_{ij} = a_i \cdot b_j = \mathrm{A}(a_i) \cdot \mathrm{A}(b_j) = (a_i + \Sigma n_k b_k) \cdot \mathrm{A}(b_j) = a_i \cdot A(b_j)$. This implies that $\mathrm{A}(b_j) = b_j$ and therefore $\mathrm{Y} = \mathrm{I}_g$. It follows that g^3 induces the identity on $H_1(\partial V; \mathbb{Z}/3)$, and g^5 induces the identity on $H_1(\partial V; \mathbb{Z}/5)$. Since $\langle g|_{\partial V} \rangle$ has finite order, a result of Serre [34] (see Corollary 4.15.15 of [41]) shows that $\langle g^3|_{\partial V} \rangle = \langle g^5|_{\partial V} \rangle = \langle 1_{\partial V} \rangle$, so $\langle g|_{\partial V} \rangle = \langle 1_{\partial V} \rangle$.

When $\partial_0 V$ is nonempty, choose a basis for $H_1(F)$ which is a union of bases from the incompressible boundary components of V together with a basis as above for the part of F, if any, that bounda a handlebody portion of V. The rest of the argument is similar to the handlebody case. This completes the proof of Lemma 6.2.

The proof of Theorem 6.1 will use the following result, for which there are variations and generalizations due to many authors. Convenient references for this version are [5, 35].

Theorem 6.3. *Suppose a group Γ acts cellularly on a contractible CW-complex L. If the 2-skeleton of L/Γ is finite, the stabilizers of the 0-cells of L are finitely presented, and the stabilizers of the 1-cells of L are finitely generated, then Γ is finitely presented. If L/Γ is finite, and the stabilizer of each cell of L is of type FL, then Γ is of type FL.*

From §5, the disc complex $L(V)$ is a contractible simplicial complex on which $\mathscr{G}(V)$ acts simplicially.

Lemma 6.4. *The quotient $L(V)/\mathscr{G}(V)$ is finite.*

Proof. Since $\mathscr{G}(V)$ has finite index in the subgroup of $\mathscr{H}(V)$ consisting of mapping classes whose restriction to $\partial_0(V)$ is isotopic to the identity, it suffices to show that up to homeomorphism fixing $\partial_0 V$, there are only finitely many collections of disjoint pairwise nonisotopic essential compressing discs in V. One proof of this proceeds as follows. View V as a 3-ball B to which 1-handles and product I-bundles over closed connected surfaces have been attached. An essential compressing disc is

called "simple" if it is contained in B. Up to homeomorphism $(\mathrm{rel}\,\partial_0 V)$, there are only finitely many simple discs. Use "slide" homeomorphisms as in [28, Lemma 3.1.1] to move a disc in a given collection to one of a finite number of choices of simple disc. This can be done so as to fix $\partial_0 V$ and also any finite collection of discs contained in ∂V. Each component of the result of cutting V along the simple disc is a product-with-handles which contains fewer discs of the collection. By induction, there are only finitely many such collections up to homeomorphism fixing the incompressible boundary components and the copies of the cut-open discs in the boundary, and the result follows.

A second approach is to establish an isomorphism of the set of collections of discs modulo the action of $\mathscr{H}(V \mathrm{\ rel\ } \partial_0 V)$ with a set of weighted graphs whose edges represent the discs in the collection and whose vertices represent the products-with-handles that result from cutting V along these discs. This approach is used in [10].

We will prove Theorem 6.1 using Theorem 6.3 and an induction that will establish that the stabilizer of each simplex of L is finitely presented and of type FL. Let $\sigma = \langle [D_0], [D_1], \cdots, [D_n] \rangle$ be an arbitrary n-simplex.

Proposition 6.5. *The natural homomorphism* $\mathscr{G}(V, D_0, D_1, \cdots, D_n)$ $\to \mathscr{G}(V)$ *is an isomorphism onto the stabilizer of* σ.

Proof. The proof begins with a technical lemma.

Lemma 6.6. *Let* $\langle h \rangle \in \mathscr{G}(V)$, *and let* D *be an essential compressing disc in* V.

(a) *If* $h(D)$ *is disjoint from* D, *then* $h(D)$ *is isotopic to* D.

(b) *If* $h(D) = D$, *then* h *does not interchange the sides of* D.

Proof of Lemma 6.6. Suppose $h(D)$ is disjoint from D and not isotopic to D. If D separates V, then $D \cup h(D)$ separates V into three components; since D and $h(D)$ are not parallel, each component must have nonvanishing homology. But h must move one of these components to be disjoint from its interior, which is impossible since h induces the identity on $H_1(M; \mathbb{Z}/3)$. Suppose that D is nonseparating. If $D \cup h(D)$ does not separate, then there is a loop in V which intersects D once and is disjoint from $h(D)$, but this loop represents an element of $H_1(V; \mathbb{Z}/3)$ which is not fixed by h. If $D \cup h(D)$ separates, then each component of the complement has nontrivial homology so must be preserved by h. But then a loop which intersects each of D and $h(D)$ once is carried to a loop which intersects each with the opposite orientation, again a contradiction. This completes the proof of (a). Part (b) is easily proved by similar considerations.

To prove Proposition 6.5, we first argue that the homomorphism is injective. Suppose that $\langle h \rangle \in \mathscr{G}(V, D_0, D_1, \cdots, D_n)$ and that there is an isotopy H from h to the identity homeomorphism. We must show that H may be chosen to preserve each D_i. Using Lemma 6.6(b), we may assume that h restricts to the identity map on each D_i. For each i, choose a basepoint $v_i \in \partial D_i$, and let τ_i be the trace at v_i of the restriction of H to F. Now $h_\#$ must induce conjugation by τ_i on $\pi_1(F, v_i)$, and since $h(\partial D_i) = \partial D_i$, it follows that τ_i is in the centralizer of $\langle \partial D_i \rangle$ in $\pi_1(F, v_i)$. If F is not a torus, then τ_i must be a power of $\langle \partial D_i \rangle$; if F is a torus, then V is a solid torus and the isotopy may be rechosen to achieve this condition. By properties of surfaces, this implies that H can be deformed to preserve ∂D_i for $0 \leq i \leq n$. Now an application of Laudenbach's theorem stated in §4.1 yields an isotopy from h to the identity preserving the D_i.

Clearly the image of the homomorphism lies in the stabilizer.

Suppose that $\langle h \rangle$ is in the stabilizer. We will argue by induction on the number of discs that h is isotopic to a homeomorphism that preserves each D_i. For one disc this is clear. Suppose there are $k+1$ discs. Lemma 6.6 shows that $h(D_i)$ is isotopic to D_i for each i. By induction, we may assume that $h(D_i) = D_i$ for $0 \leq i < k$. Let $F' = F - \bigcup_{i=0}^{k-1} \partial D_i$. The loop $h(\partial D_k)$ lies in F' and is isotopic in F to ∂D_k but not to D_i for $0 \leq i < k$. Therefore it is homotopic in F' to ∂D_k, and therefore isotopic in F' to ∂D_k. Using irreducibility of $M - \bigcup_{i=0}^{k-1} D_i$, we find that $h(D_k)$ is ambiently isotopic to D_k keeping all D_i, $0 \leq i < k$, fixed. This completes the proof of Proposition 6.5.

We can now begin the induction that will prove Theorem 6.1. The induction will be on the number of indecomposable factors in a free product decomposition for $\pi_1(V)$. If V is a 3-ball then $\mathscr{G}(V)$ is trivial. If V is the product $F \times I$, where F is a closed surface of positive genus, then using [39] the restriction $\mathscr{G}(F \times I) \rightarrow \mathscr{G}(F \times \{0\})$ is an isomorphism. As discussed in §2, the latter group is finitely presented and of type FL. So we shall assume that

(1) F is compressible.

(2) For any product-with-handles W such that the free product decomposition of $\pi_1(W)$ has fewer indecomposable factors than that of $\pi_1(V)$, the group $\mathscr{G}(W)$ is finitely presented and of type FL.

In order to be able to apply Theorem 6.3 to the disc complex of V, it remains to show that for each simplex $\sigma = \langle [D_0], [D_1], \cdots, [D_n] \rangle$ of L, the group $\mathscr{G}(V, D_0, D_1, \cdots, D_n)$, which we have identified with the stabilizer of σ, is finitely presented and of type FL.

Let W_1, W_2, $\cdots$, W_K be the components of V cut along the discs D_j. Each W_i is a product-with-handles. Each disc splits into two copies, and we denote by $D_{1,1}$, $D_{1,2}$, $\cdots$, D_{i,n_i} the copies that lie in W_i; these must all be contained in a single component F_i of ∂W_i. By Lemma 6.6, an element of $\mathscr{G}(V, D_0, D_1, \cdots, D_n)$ cannot permute the W_i nontrivially, so we have an exact sequence

$$1 \to K \to \mathscr{G}\left(V, D_0, D_1, \cdots, D_n\right)$$

$$\to \prod_{i=1}^{k} \mathscr{G}\left(W_i, D_{i,1}, D_{i,2}, \cdots, D_{i,n_i}\right) \to 1.$$

Now each element of K is representable by a homeomorphism which is supported in a neighborhood of $\bigcup_{j=0}^{n} D_j$, so K is generated by Dehn twists about the D_j. These twists have disjoint support, so K is abelian. By Lemma 6.2 and Proposition 6.5, $\mathscr{G}(V, D_0, D_1, \cdots, D_n)$ is torsion free, hence K is finitely generated free abelian. So it remains to prove that each $\mathscr{G}(W_i, D_{i,1}, D_{i,2}, \cdots, D_{i,n_i})$ is finitely presented and of type FL.

Let $E_{i,j}$ denote the center of $D_{i,j}$. It is clear that

$$\mathscr{G}(W_i, D_{i,1}, D_{i,2}, \cdots, D_{i,n_i}) \cong \mathscr{G}(W_i, E_{i,1}, E_{i,2}, \cdots, E_{i,n_i}).$$

We have the following fibration:

$$\text{GHomeo}\left(W_i, E_{i,1}, \cdots, E_{i,n_i}\right) \subseteq \text{GHomeo}\left(W_i\right)$$
$$\downarrow$$
$$\text{Imbeddings}\left(\left\{E_{i,1}, \cdots, E_{i,n_i}\right\}, F_i\right)$$

Letting $\mathscr{B}_s(F_i)$ denote the pure braid group on s strands for the 2-manifold F_i, we obtain from this fibration an exact sequence

$$\pi_1\left(\text{GHomeo}\left(W_i\right)\right) \to \mathscr{B}_{n_i}(F_i) \to \mathscr{G}\left(W_i, E_{i,1}, \cdots, E_{i,n_i}\right) \to \mathscr{G}\left(W_i\right) \to 0.$$

By induction, $\mathscr{G}(W_i)$ is finitely presented and of type FL. So it suffices to show that the quotient of $\mathscr{B}(F_i)$ by the image of $\pi_1(\text{GHomeo}(W_i))$ is finitely presented and of type FL.

We must recall a few fundamental facts about braid groups. Let Q_r be a collection of r distinct points in a compact manifold M, and let

$$C_{r,s} = \left\{(p_1, p_2, \cdots, p_s) \mid p_i \in M - Q_r, \; p_i \neq p_j \text{ for } i \neq j\right\}.$$

This space results from removing a subcomplex from the finite complex $\prod_{i=1}^{r+s} M$, so is homotopy equivalent to a finite CW-complex. The pure braid group $\mathscr{B}_s(M)$ is equal to $\pi_1(C_{0,s})$ and hence is finitely presented.

Suppose first that F_i is not a 2-sphere or torus. Then $\pi_1(\mathrm{GHomeo}(W_i))$ is trivial. Moreover, by [8, Corollary 2.2], $C_{0,s}$ is aspherical and hence $\mathscr{B}_s(F_i)$ is of type FL. Suppose F_i is a torus. Then by [8, Theorem 4] there is a homeomorphism of $C_{0,s}$ with $F_i \times C_{1,s-1}$ which is given explicitly and shows that the homomorphism

$$\pi_1\left(F_i\right) \cong \pi_1\left(\mathrm{GHomeo}\left(F_i\right)\right) \to \pi_1\left(C_{0,s}\right)$$

is an imbedding onto the factor $\pi_1(F_i)$ in the product $\pi_1(C_{0,s}) = \pi_1(f_i) \times \pi_1(C_{1,s-1})$. Hence the quotient of $\mathscr{B}(F_i)$ by the image of $\pi_1(\mathrm{GHomeo}(F_i))$ is isomorphic to $\pi_1(C_{1,s-1})$. By [8, Corollary 2.2], $C_{1,s-1}$ is aspherical so $\pi_1(C_{1,s-1})$ is of type FL. Finally, suppose $F_i = S^2$. This time we use the fibration

$$\mathrm{GHomeo}\left(W_i \ \mathrm{rel} \ F_i\right) \subseteq \ \mathrm{GHomeo}\left(W_i, E_{i,1}, E_{i,2}, \cdots, E_{i,n_i}\right)$$
$$\downarrow \rho$$
$$\mathrm{GHomeo}\left(F_i, E_{i,1}, E_{i,2}, \cdots, E_{i,n_i}\right).$$

Because W_i is a 3-ball, $\mathrm{GHomeo}(W_i \ \mathrm{rel} \ F_i)$ is contractible and the restriction fibration ρ induces an isomorphism from $\mathscr{G}(W_i, E_{i,1}, E_{i,2}, \cdots, E_{i,n_i})$ to $\mathscr{G}(F_i, E_{i,1}, E_{i,2}, \cdots, E_{i,n_i})$. The latter is a torsion-free subgroup of $\Gamma_0^{n_i}$, so is finitely presented and of type FL (see §2). This completes the proof of Theorem 6.1.

7. Mapping class groups of 3-manifolds with compressible boundary

We begin by recalling from [3], [28] some machinery for working with homeomorphisms when the boundary is compressible. If F is a compact compressible boundary component of an irreducible 3-manifold M, then there is a neighborhood V of F in M satisfying the following properties:

1. V is a connected codimension-zero submanifold of M such that $V \cap \partial M = F$ and $\partial V - F$ consists of incompressible surfaces in the interior of M.

2. V is a product-with-handles obtained from the product $(\partial V - F) \times I$ by attaching 1-handles to $(\partial V - F) \times \{1\}$.

3. If h is a homeomorphism of M such that $h(F) = F$, then h is isotopic to a homeomorphism h' of M such that $h'(V) = V$.

4. If h_1 and h_2 are isotopic homeomorphisms of M such that $h_1(V) = V$ and $h_2(V) = V$, then h_1 is isotopic to h_2 through homeomorphisms that preserve V.

Statements 3 and 4 can be given more succinctly as the assertion that $\mathscr{H}(M, V) \to \mathscr{H}(M, F)$ is an isomorphism [28, Theorem 4.1.3]. Consequently, $\mathscr{H}(M, V)$ is isomorphic to a subgroup of finite index in $\mathscr{H}(M)$.

We continue to denote $\partial V - F$ by $\partial_0 V$.

Theorem 7.1. *Let M be a compact irreducible 3-manifold with nonempty boundary. Then $\mathscr{H}(M)$ is a finitely presented group of type VFL.*

Proof. The proof proceeds by induction on the number of compressible boundary components of M. If ∂M is incompressible, then Corollary 4.3.5 gives the result. Otherwise, choose a compressible boundary component F of M and an incompressible neighborhood V of F. Denote the closure of $M - V$ by M'.

Let PHomeo(M, V) denote the subgroup of Homeo$_+(M, V)$ consisting of all h such that for every boundary component G of M, $h(G) = G$ and $h|_G$ induces the identity automorphism on $H_1(G; \mathbb{Z}/15)$. The path components of PHomeo(M, V) form a subgroup $\mathscr{P}(M)$ $\mathscr{P}(M)$ of finite index in $\mathscr{H}(M, V)$.

Every orientation-preserving homeomorphism of $\partial_0 V$ extends to V, and it is not difficult to see that the restriction PHomeo$(M, V) \to$ Homeo(M') has image precisely PHomeo(M'), over which it is a fibration with fiber GHomeo$(V$ rel $\partial_0 V)$. From this fibration, we obtain the exact sequence

$$\pi_1\left(\text{Homeo}\left(M'\right)\right) \to \mathscr{G}\left(V \text{ rel } \partial_0 V\right) \to \mathscr{P}(M, V) \to \mathscr{P}\left(M'\right) \to 0.$$

By induction applied to each component of M', the group $\mathscr{P}(M')$ is finitely presented and of type VFL. It remains to show that

$$\mathscr{G}(V \text{ rel } \partial_0 V) / \text{image}(\pi_1(\text{Homeo}(M')))$$

is finitely presented and of type FL.

Recall that $\pi_1(\text{Homeo}(M'))$ is free abelian. The image of each generator in $\mathscr{G}(V \text{ rel } \partial_0 V)$ is a collection of simultaneous Dehn twists about all boundary components of a component M_1' of M' which is Seifert fibered, such that the traces are all equal to a generator of the center of $\pi_1(M_1')$. (If M_1' is homeomorphic to $S^1 \times S^1 \times I$, then it will contribute two generators.) Now $\mathscr{G}(V \text{ rel } \partial_0 V)$ is a central extension

$$1 \to \pi_1\left(\text{Homeo}\left(\partial_0 V\right)\right) \to \mathscr{G}\left(V \text{ rel } \partial_0 V\right) \to \mathscr{G}(V) \to 1.$$

The image of $\pi_1(\text{Homeo}(M'))$ lies in $\pi_1(\text{Homeo}(\partial_0 V))$, so we have an extension

$$1 \to \pi_1\left(\text{Homeo}\left(\partial_0 V\right)\right) / \text{image}\left(\pi_1\left(\text{Homeo}\left(M'\right)\right)\right)$$
$$\to \mathscr{G}\left(V \text{ rel } \partial_0 V\right) / \text{image}\left(\pi_1\left(\text{Homeo}\left(M'\right)\right)\right) \to \mathscr{G}(V) \to 1.$$

By Theorem 6.1, $\mathscr{G}(V)$ is finitely presented and of type FL, while the kernel in this exact sequence is finitely generated free abelian. Therefore $\mathscr{G}(V \text{ rel } \partial_0 V) / \text{image}(\pi_1(\text{Homeo}(M')))$ is finitely presented and of type FL. This completes the proof of Theorem 7.1.

8. The mapping class group of the genus 2 handlebody

Theorem 8.1. *Let V be the orientable handlebody of genus 2. Then $\mathscr{H}(V)$ is a virtual duality group of dimension 3.*

The proof, which occupies the remainder of this section, is based on a theorem of R. Kramer which describes $\mathscr{H}_+(V)$ as a free product with amalgamation. In order to state that theorem, let $\{D_1, D_2, D_3\}$ be a set of disjoint essential 2-discs in V whose union separates V into two 3-cells; these discs are pairwise nonisotopic and individually nonseparating. Such a collection is called a *marking* in [23].

Theorem 8.2. *The mapping class group $\mathscr{H}_+(V)$ is a free product with amalgamation $\mathscr{H}_+(V, D_1 \cup D_2) *_{\mathscr{H}_+(V, D_1 \cup D_2, D_3)} \mathscr{H}_+(V, D_1 \cup D_2 \cup D_3)$.*

Theorem 8.2 is proved in [23]. Because that paper is unpublished and not readily available, we outline a proof here. It is essentially the proof given in [23], but adapted to the machinery we have developed in §5 of the present work.

Proof of Theorem 8.2. Let L' be the nonseparating disc complex defined in §5. Since V has genus 2, L' is 2-dimensional. By Theorem 5.4, L' is contractible. Let K be the first barycentric subdivision of L'. For each vertex of K that is a vertex of L', the star is homeomorphic to its star in L', hence its link is contractible by Theorem 5.5. Therefore the 1-complex T formed by removing all of the open stars of the vertices in K that are vertices of L' is contractible. It is invariant under the action of $\mathscr{H}_+(V)$.

Each edge of T has one endpoint which represents the isotopy class of a marking and one endpoint which represents two of the discs in that marking. It is not difficult to see that any two markings are equivalent under an orientation-preserving homeomorphism, and that the homeomorphisms that preserve the marking can achieve any permutation of the three discs in the marking. It follows that the quotient of T by the action of $\mathscr{H}_+(V)$ is a single edge. To fix notation, let $B_1 = \langle [D_1], [D_2] \rangle$ and $B_2 = \langle [D_1], [D_2], [D_3] \rangle$. Then $\langle B_1, B_2 \rangle$ is an edge of T. Using Proposition 6.5, we identify the stabilizers of $\langle B_1 \rangle$, $\langle B_2 \rangle$, and $\langle B_1, B_2 \rangle$. By the theory of groups acting on trees due to Bass and Serre [36], we obtain a

free product with amalgamation decomposition

$$\mathscr{H}_+(V) = \mathscr{H}_+(V, D_1 \cup D_2) *_{\mathscr{H}(V, D_1 \cup D_2 \cup D_3, D_1 \cup D_2)} \mathscr{H}_+(V, D_1 \cup D_2 \cup D_3)$$

$$= \mathscr{H}_+(V, D_1 \cup D_2) *_{\mathscr{H}(V, D_1 \cup D_2, D_3)} \mathscr{H}_+(V, D_1 \cup D_2 \cup D_3).$$

This completes the proof of Theorem 8.2.

Recall that $\mathscr{G}(V)$ is the subgroup of $\mathscr{H}_+(V)$ that acts trivially on $H_1(V; \mathbb{Z}/15)$. $\mathscr{G}(V)$ is a normal subgroup of finite index, and is torsion-free by Lemma 6.2.

For simplicity of notation, write $G = \mathscr{G}(V)$, $A = \mathscr{H}_+(V, D_1 \cup D_2)$, $C = \mathscr{H}(V, D_1 \cup D_2, D_3)$, and $B = \mathscr{H}_+(V, D_1 \cup D_2 \cup D_3)$.

Lemma 8.3. (a) $G \cap C$ is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

(b) $G \cap B$ is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

(c) $G \cap A$ is a duality group of dimension 3.

Proof. For (a), Lemma 6.6 shows that any element of $G \cap C$ must preserve each D_i and cannot reverse the sides, hence any element of G is isotopic to a homeomorphism supported in a neighborhood of $\bigcup_{i=1}^3 D_i$. Therefore the Dehn twists about these three discs generate G. The group they generate is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$; this is proved by restricting to ∂V. Clearly $G \cap B = G \cap C$ and (b) is proved. For (c), splitting V along $D_1 \cup D_2$ induces a surjective homomorphism from $G \cap A$ to a subgroup of finite index in $\mathscr{H}_+(D^3, D_{1,1}, D_{1,2}, D_{2,1}, D_{2,2})$, where $D_{1,1}, D_{1,2}, D_{2,1}$, and $D_{2,2}$ are disjoint 2-discs in the boundary of the 3-ball D^3. This is isomorphic to the orientation-preserving mapping class group of the 2-sphere fixing four points. By [14, Theorem 4.1], this is a virtual duality group of dimension 1. The kernel of the homomorphism is the subgroup of $G \cap A$ generated by Dehn twists about D_1 and D_2, hence is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Therefore $G \cap A$ is a duality group of dimension 3. This completes the proof of Lemma 8.3.

Regard $\mathscr{H}_+(V) = A *_C B$ as the fundamental group of the graph of groups having one edge with edge group C and two vertices with vertex groups A and B. Now G is the fundamental group of a graph of groups Γ which is a regular covering of this one-edge graph. By Lemma 6.2, G is torsion free. The restriction from $\mathscr{H}(V)$ to $\mathscr{H}(\partial V)$ is easily seen to be injective. From [14, Theorem 4.1], $\mathscr{H}(\partial V)$ is a virtual duality group of dimension 3, hence the cohomological dimension of any torsion free subgroup of $\mathscr{H}(V)$ is at most 3.

Since G is normal, every edge group of Γ is isomorphic to $G \cap C \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and every vertex group is isomorphic either to $G \cap A$ or to $G \cap B$, hence each vertex group is a duality group of dimension 3. Inductively applying [1, Proposition 9.16(a)] shows that G is a duality group

of dimension 3, hence $\mathscr{H}(V)$ is a virtual duality group of dimension 3. This completes the proof of Theorem 8.1.

9. Calculations of virtual cohomological dimension

Denote the virtual cohomological dimension of a group G by $\dim(G)$. In Lemma 3.1.1, we gave $\dim(\mathscr{H}(F, \underline{f}))$ for all 2-manifolds of finite type with compact boundary. In this section we will calculate the virtual cohomological dimension of the mapping class groups for many 3-manifolds.

We will first treat the case of compact orientable irreducible sufficiently large 3-manifolds with incompressible boundary (actually, more generally, the case when M has a complete and useful boundary pattern $\underline{m}$; recall from §3.1 that ∂M is incompressible exactly when the particular complete boundary pattern $\underline{\partial M}$ is useful). The first three propositions handle the extreme cases when the characteristic submanifold is all of M (i.e., when M is fibered) or when it is empty; in the fibered case the dimension of the mapping class group is determined from results of §3, while in the case of empty characteristic submanifold the mapping class group is finite, by a theorem of Johannson, so it has dimension 0. The main result, Theorem 9.4, gives a complicated formula for calculating $\dim(\mathscr{H}(M, \underline{m}))$. Recall from Lemma 4.2.2 that (apart from the exceptional case when M has a *Sol* structure) there is an exact sequence

$$1 \to K \to \mathscr{H}\left(M, \Sigma_1, \cdots, \Sigma_n, \underline{m}\right) \to \prod \mathscr{G}\left(\Sigma_i, \underline{\sigma_i}; \underline{\mathrm{Fr}(\Sigma_i)}\right) \to 1,$$

where $\mathscr{H}(M, \Sigma_1, \cdots, \Sigma_n, \underline{m})$ is isomorphic to a subgroup of finite index in $\mathscr{H}(M, \underline{m})$, K is the finitely generated abelian subgroup of $\mathscr{H}(M, \underline{m})$ generated by Dehn twists about the components of the frontier of the characteristic submanifold Σ, and for each component Σ_i of Σ, $\mathscr{G}(\Sigma_i, \underline{\sigma_i}; \mathrm{Fr}(\Sigma_i))$ is the subgroup of $\mathscr{H}(\Sigma_i, \underline{\sigma_i})$ consisting of the mapping classes whose restrictions to $\mathrm{Fr}(\Sigma_i)$ are isotopic to the identity. The virtual cohomological dimension (i.e., the rank) of K is calculated in Lemma 9.5; one takes the abstract free abelian group generated by a set of Dehn twists about the components of the frontier of Σ which generate K, then divides out by the relations which derive from the circular isotopies of the complementary pieces. In Lemma 9.6, the dimensions of the $\mathscr{G}(\Sigma_i, \underline{\sigma_i}; \underline{\mathrm{Fr}(\Sigma_i)})$ are calculated. In the I-bundle case, this subgroup has finite index in $\mathscr{H}(\Sigma_i, \underline{\sigma_i})$, so from §3.2 its dimension equals $\dim(\mathscr{H}(F_i, \underline{f_i}))$, where F_i

is as usual the base surface of the I-bundle. The Seifert-fibered case is reduced, using the exact sequence of Theorem 3.5.2, to calculating the rank of the abelian group $\mathscr{G}(\Sigma_i, \underline{\sigma}_i; \mathrm{Fr}(\Sigma_i)) \cap \mathscr{H}^0(\Sigma_i, \underline{\sigma}_i)$. This is the group generated by vertical Dehn twists excluding those about vertical annuli with one boundary component in a torus component of $\mathrm{Fr}(\Sigma_i)$, so can be expressed, using the correspondence of Lemma 3.5.3, in terms of a homology group of the base surface. Combining the two lemmas gives the formula in Theorem 9.4.

In the remainder of the section, we obtain some estimates for the virtual cohomological dimension of the mapping class groups of products-with-handles (see §6), including the special case of handlebodies. If V is a product-with-handles, then the restriction $\mathscr{H}(V) \to \mathscr{H}(\partial V)$ is injective, so Harer's calculation of $\dim(\mathscr{H}(F))$ (see §2) gives upper bounds for $\dim(\mathscr{H}(V))$. Lower bounds can be obtained inductively by building up a subgroup $\mathscr{H}(V, D)$ as a sequence of extensions, where D is a compressing disc. Since $\mathscr{H}(V, D)$ is a subgroup of $\mathscr{H}(V)$, by Proposition 6.5, this provides the lower bounds for $\dim(\mathscr{H}(V))$.

For I-bundles, we have the following immediate consequence of Proposition 3.2.1.

Proposition 9.1. *Let $(\Sigma, \underline{\sigma})$ be an irreducible I-bundle over $(F, \underline{f})$. Then $\dim(\mathscr{H}(\Sigma, \underline{\sigma})) = \dim(\mathscr{H}(F, \underline{f}))$.*

Suppose now that $(\Sigma, \underline{\sigma})$ is Seifert fibered over $(F, \underline{f})$. Let $\mathscr{E}$ be the exceptional points in F. Define F' to be $F - \mathscr{E}$ and regard $\underline{f}$ as a boundary pattern on F'.

Proposition 9.2. *Let $(\Sigma, \underline{\sigma})$ be a compact orientable irreducible sufficiently large 3-manifold which is Seifert fibered over $(F, \underline{f})$.*

(a) *Let Σ be an S^1-bundle over the disc. If some component of $\underline{\sigma}$ is an annulus, then $\dim(\mathscr{H}(\Sigma, \underline{\sigma})) = 0$, otherwise $\dim(\mathscr{H}(\Sigma, \underline{\sigma})) = 1$.*

(b) *Let Σ be an S^1-bundle over the annulus. If both components of $\partial\Sigma$ contain an element of $\underline{\sigma}$ which is an annulus, then $\dim(\mathscr{H}(\Sigma, \underline{\sigma})) = 2$, otherwise $\dim(\mathscr{H}(\Sigma, \underline{\sigma})) = 1$.*

(c) *Let Σ be an S^1-bundle over the torus. If the Euler class is zero, then $\dim(\mathscr{H}(\Sigma)) = 3$, otherwise $\dim(\mathscr{H}(\Sigma)) = 1$.*

(d) *Let Σ be an S^1-bundle over the Klein bottle. If the Euler class is zero, then $\dim(\mathscr{H}(\Sigma)) = 1$, otherwise $\dim(\mathscr{H}(\Sigma)) = 0$.*

(e) *Otherwise, $\dim(\mathscr{H}(\Sigma, \underline{\sigma})) = \mathrm{rank}(H_1(F, \partial F)) + \dim(\mathscr{H}(F', \underline{f}))$.*

Proof. Let $\underline{\sigma}_1$ be the set of components of $\partial\Sigma - |\underline{\sigma}|$, so that $\underline{\sigma} \cup \underline{\sigma}_1$ is a boundary pattern for Σ. Now $\mathscr{H}(\Sigma, \underline{\sigma} \cup \underline{\sigma}_1)$ is a subgroup of finite index

in $\mathscr{H}(\Sigma, \underline{\sigma})$ (the subgroup that permutes the elements of $\underline{\underline{\sigma_1}}$ trivially), so upon replacing $\underline{\sigma}$ by $\underline{\sigma} \cup \underline{\underline{\sigma_1}}$, we may assume that $\underline{\sigma}$ is a complete boundary pattern.

If Σ is a solid torus and $\underline{\sigma}$ does not contain an annulus, then a Dehn twist about the essential compressing disc generates an infinite cyclic subgroup of finite index, while if $\underline{\sigma}$ does contain an annulus, the mapping class group is finite (this can be proved by geometric methods, or by use of Theorems 3.5.1 and 3.5.2 and Lemma 3.5.3). This gives (a). Parts (b), (c), and (d) are Propositions 3.4.1, 3.4.3, and 3.4.4 respectively. If Σ is the S^1-bundle over the Möbius band, then Proposition 3.4.2 shows $\dim(\mathscr{H}(\Sigma, \underline{\sigma})) = 0$, which agrees with the value in (e). If Σ fibers over the 3-sphere with three exceptional orbits, then by Proposition 3.4.5, its mapping class group is finite. Since the mapping class group of the thrice-punctured sphere is finite, the formula in (e) yields the correct dimension. If M is the Hantsche-Wendt manifold (see Theorem 3.5.1) then $\mathscr{H}(M)$ is finite [6] and the value given by (e) is again correct. For the remainder of the proof, we may assume that Σ is not one of the exceptions 5.1.1–5.1.5 of [21], that Σ is not an S^1-bundle over the annulus, Möbius band, torus, or Klein bottle, and that Σ is not the Hantsche-Wendt manifold. In particular, $\mathscr{H}(\Sigma, \underline{\sigma}) \cong \mathscr{H}^f(\Sigma, \underline{\sigma})$, by Theorem 3.5.1.

Recall the groups $\mathscr{H}^0(\Sigma, \underline{\sigma})$ and $\mathscr{H}(F, \underline{f})$ defined just before Theorem 3.5.2. Now $\mathscr{H}^0(\Sigma, \underline{\sigma})$ is isomorphic to $H_1(F, \partial F)$, by Lemma 3.5.3, while $\mathscr{H}^*(F, \underline{f})$ is isomorphic to a subgroup of finite index in $\mathscr{H}(F', \underline{f})$, hence is a virtual duality group. These are related by the exact sequence

$$1 \to \mathscr{H}^0\left(\Sigma, \underline{\underline{\sigma}}\right) \to \mathscr{H}_+^f\left(\Sigma, \underline{\underline{\sigma}}\right) \to \mathscr{H}^*\left(F, \underline{\underline{f}}\right) \to 1$$

of Theorem 3.5.2. By Lemma 3.5.9, there is a subgroup (hence a normal subgroup) of finite index in $\mathscr{H}_+^f(\Sigma, \underline{\sigma})$ which intersects $\mathscr{H}^0(\Sigma, \underline{\sigma})$ in a finitely generated free abelian group, and applying Lemma 1.4(c) to this subgroup shows that

$$\dim\left(\mathscr{H}\left(\Sigma, \underline{\underline{\sigma}}\right)\right) = \dim\left(\mathscr{H}^0\left(\Sigma, \underline{\underline{\sigma}}\right)\right) + \dim\left(\mathscr{H}^*\left(F, \underline{\underline{f}}\right)\right)$$

$$= \mathrm{rank}\left(H_1\left(F, \partial F\right)\right) + \dim\left(\mathscr{H}\left(F', \underline{\underline{f}}\right)\right)$$

This completes the proof of Proposition 9.2.

When the characteristic submanifold of $(M, \underline{m})$ is empty, $\mathscr{H}(M, \underline{m})$ is finite by [21, Theorem 27.1]. Therefore we have

Proposition 9.3. *If the characteristic submanifold of* $(M, \underline{m})$ *is empty, then* $\dim(\mathscr{H}(M, \underline{m})) = 0$.

We now require one more definition. Let (Σ_i, σ_i) be a component of the characteristic submanifold of a 3-manifold, and let $(F_i, \underline{f_i})$ be the orbit surface. Denote by $\widehat{F}_i$ the 2-manifold that results from capping off with a 2-disc each boundary component of ∂F_i that is the image of a component of $\mathrm{Fr}(\Sigma_i)$ that is a torus. The next theorem includes the remaining Haken cases.

Theorem 9.4. *Let* $(M, \underline{m})$ *be a compact orientable irreducible sufficiently large 3-manifold with complete and useful boundary pattern. Suppose that the characteristic submanifold* Σ *of* $(M, \underline{m})$ *is not empty and not equal to* M. *Let* (Σ_1, σ_1), (Σ_2, σ_2), $\ldots$, (Σ_n, σ_n) *be the components of* Σ, *and let* $(S_1, \underline{s_1})$, $(S_2, \underline{s_2})$, $\ldots$, $(S_m, \underline{s_m})$ *be the components of the closure of* $M - \Sigma$. *Let* t *be the number of components of* $\mathrm{Fr}(\Sigma)$ *that are tori and let* a *be the number that are annuli. Then*

$$\dim\left(\mathscr{H}\left(M, \underline{\underline{m}}\right)\right) = 2t + a + k - \sum_{i=1}^{n} \mathrm{rank}\left(\mathrm{center}\left(\pi_1\left(\Sigma_i\right)\right)\right)$$

$$- \sum_{j=1}^{m} \mathrm{rank}\left(\mathrm{center}\left(\pi_1\left(S_j\right)\right)\right)$$

$$+ \sum_{i=1}^{n} \dim\left(\mathscr{H}\left(F_i', \underline{\underline{f_i}}\right)\right)$$

$$+ \sum_{\{i | \Sigma_i \text{ is Seifert fibered}\}} \mathrm{rank}\left(H_1\left(\widehat{F}_i, \partial \widehat{F}_i\right)\right),$$

where k *is the number of components of* Σ *that are homeomorphic to* $S^1 \times S^1 \times I$ *and contain a component of the frontier of* Σ *which is an annulus.*

Proof. By hypothesis Σ is nonempty and not equal to M; in particular each component of Σ has nonempty boundary.

Suppose first that M is a torus bundle over S^1 that admits a *Sol* structure. In this case the characteristic submanifold $\Sigma = \Sigma_1 = S^1 \times S^1 \times I$ is a regular neighborhood of a torus fiber, and is Seifert-fibered over the annulus $F_1 = F_1'$, and S_1 is also homeomorphic to $S^1 \times S^1 \times I$. In the notation of the statement of Theorem 9.4, we have $t = 2$, $a = 0$, $k = 0$, $\mathrm{rank}(\mathrm{center}(\pi_1(\Sigma_1))) = 2$, $\mathrm{rank}(\mathrm{center}(\pi_1(S_1))) = 2$, $\dim(\mathscr{H}(F_1')) = 0$, and $\mathrm{rank}(H_1(\widehat{F}_1, \partial \widehat{F}_1)) = 0$, since $\widehat{F}_1$ is a 2-sphere. The formula in Theorem 9.4 yields $\dim(\mathscr{H}(M, \underline{m})) = 0$, which agrees with Proposition

4.1.2. (What is really going on in this case is that the homeomorphism h defined by performing the monodromy on each torus fiber generates an infinite cyclic subgroup of finite index in $\mathcal{H}(M, \Sigma)$, while h is isotopic to the identity if Σ is not required to be preserved, so $\mathcal{H}(M)$ is finite. But h and its power are not fiber-preserving with respect to the Seifert fibering of Σ, and the formula in Theorem 9.4 only detects those classes in $\mathcal{H}(M, \Sigma, \underline{m})$ which are fiber-preserving on Σ. Apart from this exceptional case, these are all the mapping classes.)

From now on, we assume that $(M, \underline{m})$ is not one of these manifolds, and hence by Proposition 4.1.1 that $\mathcal{\overline{\overline{H}}}(M, \underline{m}) \cong \mathcal{H}(M, \Sigma, \underline{m})$. By Lemma 4.2.1, the kernel $\mathcal{H}(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m})$ of the restriction

$$\mathcal{H}\left(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m}\right) \to \mathcal{H}\left(\overline{M - \Sigma}, \underline{\underline{Fr(\Sigma)}}\right)$$

has finite index, and by Lemma 4.2.2, there is a restriction homomorphism

$$\rho: \mathcal{H}\left(M, \Sigma_1, \Sigma_2, \cdots, \Sigma_n, \underline{m}\right) \to \prod \mathcal{G}\left(\Sigma_i, \underline{\sigma_i}; \underline{\underline{Fr(\Sigma_i)}}\right)$$

which is surjective, and whose kernel K is the subgroup generated by the Dehn twists about the components of the frontier of Σ. Therefore Theorem 9.4 will be an immediate consequence of the next two lemmas. The reader may wish to review Example 4.2.4 before examining Lemma 9.5.

Lemma 9.5. *The rank of K is*

$$2t + a + k - \sum_{i=1}^{n} \text{rank}\left(\text{center}\left(\pi_1\left(\Sigma_i\right)\right)\right) - \sum_{j=1}^{n} \text{rank}\left(\text{center}\left(\pi_1\left(S_j\right)\right)\right),$$

where k is the number of components of Σ that are homeomorphic to $S^1 \times S^1 \times I$ and contain a component of the frontier of Σ which is an annulus.

Proof. For each torus component of $\text{Fr}(\Sigma)$, choose generators z_i and w_i for the fundamental group so that z_i represents the fiber and w_i represents a cross section to the fibering. Choose Dehn twists with traces z_i and w_i respectively, supported in a neighborhood of the torus. For each annulus component of $\text{Fr}(\Sigma)$, choose a Dehn twist supported in a neighborhood of the annulus. Let K_1 be the abstract free abelian group of rank $2t + a$ on the chosen set of Dehn twists. Passing to isotopy classes in $\mathcal{H}(M, \underline{m})$ defines a surjective homomorphism from K_1 to K. To prove Lemma 9.5, we will show that the rank of the kernel of this homomorphism is $\Sigma_{i=1}^{n} \text{rank}(\text{center}(\pi_1(\Sigma_i))) + \Sigma_{j=1}^{m} \text{rank}(\text{center}(\pi_1(S_j))) - k$.

Choose basepoints u_i in the interior of the Σ_i and v_j in the interior of the S_j. If either $\Sigma_i = S^1 \times S^1 \times I$ and some component of $\mathrm{Fr}(\Sigma_i)$ is an annulus, or Σ_i is an S^1-fibered solid torus (in which case the components of $\mathrm{Fr}(\Sigma_i)$ must be annuli), defined $\mathscr{Z}(\pi_1(\Sigma_i, u_i))$ to be the subgroup of $\pi_1(\Sigma_i, u_i)$ generated by the fiber. For all other Σ_i, define $\mathscr{Z}(\pi_1(\Sigma_i, u_i))$ to be the center of $\pi_1(\Sigma_i, u_i)$. The rank of $\mathscr{Z}(\pi_1(\Sigma_i, u_i))$ equals the rank of the center of $\pi_1(\Sigma_i, u_i)$ except when $\Sigma_i = S^1 \times S^1 \times 1$ and some component of $\mathrm{Fr}(\Sigma_i)$ is an annulus, in which case the center has rank 2 while $\mathscr{Z}(\pi_1(\Sigma_i))$ has rank 1. This accounts for the term k in the formula in Lemma 9.5. Define $\mathscr{Z}(\pi_1(S_j, v_j))$ to be the center of $\pi_1(S_j, v_j)$. The only S_j whose fundamental groups have nontrivial center are I-bundles over the annulus or torus (see [21, p. 159]). Observe that the elements of $\mathscr{Z}(\pi_1(\Sigma_i, u_i))$ and $\mathscr{Z}(\pi_1(S_j, v_j))$ are precisely the possible traces of admissible circular isotopies (a circular isotopy is an isotopy that starts and ends at the identity) of Σ_i or S_j. Define a homomorphism

$$\phi \colon \prod_{i=1}^{n} \mathscr{Z}\left(\pi_1\left(\Sigma_i, u_i\right)\right) \times \prod_{j=1}^{m} \mathscr{Z}\left(\pi_1\left(S_j, v_j\right)\right) \to K_1$$

on generators by sending $x \in \mathscr{Z}(\pi_1(\Sigma_i, u_i))$ to the product of Dehn twists about all the components of $\mathrm{Fr}(\Sigma_i)$ such that each has trace homotopic to x, and similarly for $y \in \mathscr{Z}(\pi_1(S_j, v_j))$. There is an isotopy (rel $\mathrm{Fr}(\Sigma_i)$) from each such product to the identity, whose trace at u_i or v_j is x or y. We will complete the proof by showing that ϕ is injective and has image equal to the kernel of the homomorphism from K_1 to K.

From an element $(x_1, \cdots, x_n, y_1, \cdots, y_m)$ of the kernel of ϕ, one obtains a circular isotopy of $(M, \underline{m})$ whose trace at each u_i is x_i and at each v_i is y_i. Since $(M, \underline{m})$ is aspherical, the trace of any circular isotopy must be central. But the fundamental group of M is centerless, since M is not Seifert fibered or I-fibered, and therefore ϕ is injective.

Suppose that w is an element of K_1 that maps to the trivial element of K. Then w is a product of Dehn twists about frontier components of Σ that is isotopic to the identity preserving Σ. Let H be such an isotopy. The traces of H at the basepoints u_i and v_j yield an element w_1 of $\prod_{i=1}^{n} \mathscr{Z}(\pi_1(\Sigma_i, u_i)) \times \prod_{j=1}^{m} \mathscr{Z}(\pi_1(S_j, v_j))$. As a homeomorphism of M, ww_1^{-1} is isotopic to the identity by an isotopy which fixes all basepoints and preserves each Σ_i. It follows that $w = w_1$ as elements of K_1. This completes the proof of Lemma 9.5.

Lemma 9.6. (a) If $(\Sigma_i, \underline{\underline{\sigma_i}})$ is an I-bundle, then $\dim(\mathscr{G}(\Sigma_i, \underline{\underline{\sigma_i}}; \mathrm{Fr}(\Sigma_i)))$
$= \dim(\mathscr{H}(F_i', f_i))$.

(b) If $(\Sigma_i, \underline{\underline{\sigma_i}})$ is Seifert fibered, then

$$\dim\left(\mathscr{G}\left(\Sigma_i, \underline{\underline{\sigma_i}}; \mathrm{Fr}\left(\Sigma_i\right)\right)\right) = \mathrm{rank}\left(H_1\left(\widehat{F}_i, \partial\widehat{F}_i\right)\right) + \dim\left(\mathscr{H}\left(F_i', \underline{\underline{f_i}}\right)\right).$$

Proof. The group $\mathscr{G}(\Sigma_i, \underline{\underline{\sigma_i}}; \mathrm{Fr}(\Sigma_i))$ is by definition the normal subgroup of $\mathscr{H}(\Sigma_i, \underline{\underline{\sigma_i}})$ consisting of the mapping classes whose restrictions to each component of $\mathrm{Fr}(\Sigma_i)$ are isotopic to the identity. Suppose first that Σ_i is an I-bundle, so $F_i' = F_i$. Since the components of $\mathrm{Fr}(\Sigma_i)$ are squares and/or annuli, their mapping classes are finite, and therefore $\mathscr{G}(\Sigma_i, \underline{\underline{\sigma_i}}; \mathrm{Fr}(\Sigma_i))$ has finite index in $\mathscr{H}_+(\Sigma_i, \underline{\underline{\sigma_i}})$. By Proposition 3.2.1, the latter is isomorphic to $\mathscr{H}_+(F_i, f_i)$, so Lemma 9.6(a) follows. Suppose that $(\Sigma_i, \underline{\underline{\sigma_i}})$ is Seifert-fibered. In Lemma 3.6.2, it is shown that in the exact sequence from Theorem 3.5.2,

$$1 \to \mathscr{H}^0\left(\Sigma, \underline{\sigma}\right) \to \mathscr{H}_+^f\left(\Sigma, \underline{\sigma}\right) \to \mathscr{H}^*\left(F, \underline{f}\right) \to 1,$$

the image of $\mathscr{G}(\Sigma_i, \underline{\underline{\sigma_i}}; \mathrm{Fr}(\Sigma_i))$ in $\mathscr{H}^*(F, \underline{f})$ has finite index. To determine $\mathscr{G}(\Sigma_i, \underline{\underline{\sigma_i}}; \mathrm{Fr}(\Sigma_i)) \cap \mathscr{H}^0(\Sigma_i, \underline{\underline{\sigma_i}})$, consider the restriction of $\mathscr{H}^0(\Sigma_i, \underline{\underline{\sigma_i}})$ to the components of $\mathrm{Fr}(\Sigma_i)$. The restriction to each annulus is isotopic to the identity, but the restriction to each torus is a vertical homeomorphism of the fibered torus, which is a product of Dehn twists about the fiber. Let $T_1, \cdots, T_k$ be the torus components of $\mathrm{Fr}(\Sigma_i)$, and let $B_1, \cdots, B_k$ be their images in ∂F_i. Now $\mathscr{G}(\Sigma_i, \underline{\underline{\sigma_i}}; \mathrm{Fr}(\Sigma_i)) \cap \mathscr{H}^0(\Sigma_i, \underline{\underline{\sigma_i}})$ corresponds to the kernel of the restriction from $\mathscr{H}^0(\Sigma_i, \underline{\underline{\sigma_i}})$ to $\prod \mathscr{H}(T_i)$. Under the correspondence of Lemma 3.5.3, this restriction becomes

$$H_1\left(F_i, \partial F_i\right) \xrightarrow{\partial} H_0\left(\partial F_i\right) \to \Pi_{j=1}^k H_0\left(B_i\right),$$

where the second homomorphism is projection. The kernel of the homomorphism is isomorphic to $H_1(\widehat{F}_i, \partial\widehat{F}_i)$. This completes the proof of Lemma 9.6 and hence of Theorem 9.4.

We turn now to the case of compressible boundary. Let $\{D_1, D_2, \cdots, D_s, E_1, \cdots, E_r\}$ be a (possibly empty) collection of pairwise disjoint 2-discs in the boundary of V. Denote by $H_{g,r}^s$ the mapping class group $\mathscr{H}_+(V, D_1, D_2, \cdots, D_s \text{ rel } E_1 \cup E_2 \cup \cdots \cup E_r)$. We will need the following lemma, which is a special case of [1, Theorem 5.6].

Lemma 9.7. *Let* $1 \to N \to G \to Q \to 1$ *be a short exact sequence of groups. Assume that* N *is of type FL, and that* $H^n(N; \mathbb{Z}N)$ *is* $\mathbb{Z}$*-free for* n *equal to the cohomological dimension of* N*. If the cohomological dimension of* Q *is finite, then* $\dim(G) = \dim(N) + \dim(Q)$*.*

For the case of handlebodies, we have the following estimate for the virtual cohomological dimension.

Theorem 9.8. *Let* V *be a handlebody of genus* $g \geq 2$*.*

(a) *If* $g = 2$*, then* $\dim(H^0_{2,0}) = 3$*, and* $\dim(H^s_{2,r}) = 4 + 2r + s$ *if* $r + s > 0$*.*

(b) *If* $g \geq 3$*, then* $3g - 2 \leq \dim(H^0_{g,0}) \leq 4g - 5$*, and* $3g + 2r + s - 2 \leq \dim(H^s_{g,r}) \leq 4g + 2r + s - 4$ *if* $r + s > 0$*.*

Proof. There is a restriction homomorphism from $H^s_{g,r}$ to $\mathscr{H}_+(\partial V, D_1, D_2, \cdots, D_s \text{ rel } E_1 \cup E_2 \cup \cdots \cup E_r)$. It is not difficult to prove that this homomorphism is injective: if a homeomorphism of V is the identity on ∂V, then using irreducibility of V it is isotopic to the identity on the union of ∂V and a collection of compressing discs that cut V into a 3-ball, and the Alexander trick furnishes an isotopy to the identity. Since

$$
\begin{aligned}
&\mathscr{H}_+\left(\partial V, D_1, D_2, \cdots, D_s \text{ rel } E_1 \cup E_2 \cup \cdots \cup E_r\right) \\
&\cong \mathscr{H}_+\left(\partial V - (\text{int}(E_1) \cup \cdots \cup \text{int}(E_r)), \right. \\
&\qquad\qquad \left. D_1, D_2, \cdots, D_s \text{ rel } \partial E_1 \cup \cdots \cup \partial E_r\right) \\
&\cong \Gamma^s_{g,r},
\end{aligned}
$$

Harer's theorem (see §2) provides the upper bounds given in Theorem 9.8.

Since $H^s_{g,r}$ is an extension of $\mathbb{Z}^r$ by $H^{r+s}_{g,0}$, Lemma 9.7 shows that $\dim(H^s_{g,r}) = \dim(H^{r+s}_{g,0}) + r$. Therefore it suffices to consider only the case when $r = 0$.

We first consider the case of $g = 2$. By Theorem 8.1, $H^0_{2,0}$ is a virtual duality group of dimension 3.

Let d_i be a point in the interior of D_i. Observe that

$$
H^s_{g,0} \cong \mathscr{H}_+(V \text{ rel } \{d_1, \cdots, d_s\}).
$$

The restriction fibration from $\text{Homeo}_+(V)$ to $\text{Imb}(\{d_1\}, \partial V)$ yields an exact sequence

$$
1 \to \pi_1\left(\text{Imb}\left(\{d_1\}, \partial V\right)\right) \to \mathscr{H}_+\left(V, \{d_1\}\right) \to H^0_{g,0} \to 1.
$$

Since $\text{Imb}(\{d_1\}, \partial V)$ is homeomorphic to ∂V, $\pi_1(\text{Imb}(\{d_1\}, \partial V))$ is a duality group of dimension 2. Therefore the exact sequence shows that $\mathscr{H}_+(V \text{ rel } d_1)$ is a virtual duality group of dimension 5. Inductively, for

$s \geq 1$, we have a similar exact sequence

$$1 \to \pi_1\left(\text{Imb}\left(\{d_{s+1}\}, \partial V - \{d_1, \cdots, d_s\}\right)\right) \to \mathscr{H}_+\left(V \text{ rel } \{d_1, \cdots, d_{s+1}\}\right)$$
$$\to \mathscr{H}_+\left(V \text{ rel } \{d_1, \cdots, d_s\}\right) \to 1.$$

This time $\text{Imb}(\{d_{s+1}\}, \partial V - \{d_1, \cdots, d_s\})$ is homeomorphic to $\partial V - \{d_1, \cdots, d_s\}$, so its fundamental group is a free group, which is a duality group of dimension 1. By induction, $\mathscr{H}_+(V \text{ rel} \{d_1, d_2, \cdots, d_s\})$ is a virtual duality group of dimension $4 + s$. It follows that $\mathscr{H}_+(V, \text{rel} \{d_1, \cdots, d_{s+1}\})$ is a virtual duality group of dimension $4 + s + 1$, which completes the induction and the case $g = 2$.

Now suppose $g \geq 3$. Let D be a nonseparating compressing disc for V. Let V' be the handlebody of genus $g-1$ that results from splitting V along D; its boundary contains two copies D' and D'' of D. Obviously,

$$\mathscr{H}_+\left(V, D_1, D_2, \cdots, D_s \text{ rel } D\right) \cong \mathscr{H}_+\left(V', D_1, D_2, \cdots, D_s \text{ rel } D' \cup D''\right),$$

so using induction on g we have

$$3g + 1 - s \leq \dim(\mathscr{H}_+(V, D_1, D_2, \cdots, D_s \text{ rel } D)).$$

From the restriction fibration from $\text{Homeo}_+(V, D, D_1, D_2, \cdots, D_s)$ to $\text{Homeo}(D)$, we obtain an exact sequence

$$1 \to \mathbb{Z} \to \mathscr{H}_+\left(V, D_1, D_2, \cdots, D_s \text{ rel } D\right)$$
$$\to \mathscr{H}_+\left(V, D, D_1, D_2, \cdots, D_s\right) \to \mathbb{Z}/2.$$

By Lemma 9.7, this shows that the virtual cohomological dimension of $\mathscr{H}_+(V, D, D_1, D_2, \cdots, D_s)$ is at least $3g + s - 2$. Since (as in Proposition 6.5) $\mathscr{H}_+(V, D, D_1, D_2, \cdots, D_s)$ is isomorphic to a subgroup of $\mathscr{H}(V, D_1, D_2, \cdots, D_s)$, these estimates give lower bounds for $\dim(H_{g,r}^s)$. This completes the proof of Theorem 9.8.

In the next theorem, let V be an orientable product-with-handles which is not a handlebody. If V is actually a product $F \times I$ (with F closed), let F_1 and F be the boundary components of V. Otherwise, let $F_1, \cdots, F_k$ be the incompressible boundary components of V, and let F be the compressible boundary component. Denote the genus of F by g. Suppose that V has $(k-1) + g_0$ 1-handles (so that $\pi_1(V)$ has g_0 infinite cyclic free factors). Let $\{D_1, D_2, \cdots, D_s, E_1, \cdots, E_r\}$ be a (possibly empty) collection of pairwise disjoint 2-discs in F. Denote by $H_{g,r}^s$ the mapping class group $\mathscr{H}_+(V, D_1, D_2, \cdots, D_s \text{ rel } E_1 \cup E_2 \cup \cdots \cup E_r)$.

Theorem 9.9. *Let V be an orientable product-with-handles, which is not a handlebody.*

(a) *Suppose* V *is a product* $F \times I$. *If* $g = 1$, *then* $\dim(H^0_{g,0}) = 1$. *If* $2g + r + s > 2$, *then* $\dim(H^0_{g,0}) = 4g - 5$ *and* $\dim(H^s_{g,r}) = 4g + 2r + s - 4$ *if* $r + s > 0$.

(b) *Suppose* V *is not a product and not a solid torus. Then* $4g - g_0 - k + 2r + s - 3 \leq \dim(H^s_{g,r}) \leq 4g - 5$.

Proof. Denote the genus of F_i by g_i. As in Theorem 9.8, the restriction to F is injective on mapping class groups, which using Harer's theorem (see §2) gives (a) and the upper bound in (b). To obtain the lower bound in (b), we will induct on g_0. As in the proof of Theorem 9.8, we may assume that $r = 0$. Since V is not a handlebody, $k > 0$.

Suppose that $g_0 = 0$. Let D be a compressing disc such that one component of the result of cutting V along D is $F_k \times I$; the other is a product-with-handles V' which has $(k - 1)$ incompressible boundary components. Let F' be the compressible boundary component of V' (or the component other than F_1, in case $k = 2$). The genus of F' is $g - g_k$. We may choose D so that all the D_i lie in V'. Now $\mathcal{H}_+(V \text{ rel } D) \cong \mathcal{H}_+(V' \text{ rel } D) \times \mathcal{H}_+(F_k \times I \text{ rel } D)$. By [14], $\mathcal{H}_+(F_k \times I \text{ rel } D) \cong \mathcal{H}_+(F_k \text{ rel } D) \cong \mathcal{H}_+(F_k - \text{int}(D) \text{ rel } \partial D)$ is a virtual duality group with $\mathbb{Z}$-free dualizing module, so Lemma 9.7 and induction give

$$\dim\left(\mathcal{H}_+\left(V, D_1, D_2, \cdots, D_s \text{ rel } D\right)\right)$$
$$= \dim\left(\mathcal{H}_+\left(V', D_1, D_2, \cdots, D_s \text{ rel } D\right)\right) + \dim\left(\mathcal{H}_+\left(F_k \times I \text{ rel } D\right)\right)$$
$$\geq 4\left(g - g_k\right) - (k - 1) + 2 + s - 3 + 4g_k - 2$$
$$= 4g - k + s - 2.$$

As in the handlebody case, this implies that $\dim(H^s_{g,0}) \geq 4g - k + s - 3$.

Suppose now that $g_0 \geq 1$. Let D be a nonseparating disc and let V' be the result of cutting V along D. The boundary of V' contains two copies D' and D'' of D. By induction on g_0, we have

$$\dim\left(\mathcal{H}_+\left(V, D_1, D_2, \cdots, D_s \text{ rel } D\right)\right)$$
$$= \dim\left(\mathcal{H}_+\left(V', D_1, D_2, \cdots, D_s \text{ rel } D' \cup D''\right)\right)$$
$$\geq 4\left(g - 1\right) - \left(g_0 - 1\right) - k + 4 + s - 3$$
$$= 4g - g_0 - k + s - 2.$$

As above, this implies that $\dim(H^s_{g,0}) \geq 4g - g_0 - k + s - 3$. This completes the induction and the proof of Theorem 9.9.

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